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APPROXIMATE SOLUTION OF LINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND BY THE METHOD OPTIMAL QUADRATURES

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The paper considers the application of the optimal quadrature formula in the space to numerical solution of linear Fredholm integral equations of the second kind. The results of specific examples are analyzed. The exact solution is used to compare the results. It is proved that as m increases, the optimal quadrature formulas in the space give high accuracy for solving the integral equation.

Keywords: linear integral equation, optimal quadrature formula, coefficients of optimal quadrature formula, absolute error.

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1 Introduction

Approximate solution of Fredholm linear integral equations of the second kind has been and is still being practiced by very many researchers in the world. For example, in [1], a modified multistage mean value integral method is applied to solve Fredholm integral equations of the second kind. The proposed algorithm is based on the application of multistage scheme to the modified mean integral value method. The effectiveness of this method is illustrated by examples.

The authors of [2] proposed a numerical method for solving integral equations using Chebyshev polynomials and Lagrange interpolation formula. They also showed results on examples with Volterra and Fredholm integral equations of the first kind.

Researchers from Morocco in their paper [3] considered the polynomial Lejandre-Kontorovich method for the numerical solution of the Fredholm integral equation of the second kind with a smooth kernel. Numerical examples are given to illustrate the theoretical estimates.

In [4] the authors presented a new semi-analytic method for solving linear and nonlinear integral and Fredholm integro-differential equations of the second kind and systems including them. The main idea of the method is to apply the mean value theorem of integrals. The possibilities of the method are demonstrated with the help of examples.

Applications of the method of Boo-Baker polynomials for approximate solution of Volterra and Fredholm integral equations of the second kind are proposed by the author of the article [5]. The author argues that these polynomials are an incredibly useful mathematical tool because they are very easily defined. Numerical results in tabular and graphical forms are given.

Russian researchers in their work [6] proposed the application of polynomial integro-differential splines of the second and third order to the solution of integral equations, in particular, to the solution of Fredholm integral equations of the second kind. An appropriate quadrature formula is proposed for computing the integral in piecewise quadratic integro-differential spline formulas. Results of numerical experiments are also given.

The paper [7] is devoted to the study of the numerical solution of the Fredholm integral equation of the second kind by the spline collocation method. The author argues that for numerical solutions of such problems the classical collocation method using polynomials in spaces of p-summable functions is not always realizable, and even in case of its realization it is not always possible to obtain characteristics and error estimates of such approximations. For the main results collocation splines of the third order, as well as integral and averaged smoothness moduli are used. At the same time the obtained results can be a starting point for works with collocation splines of higher orders.

The author of [8] proposed a new direct interpolation method for solving linear integral equations with weakly singular kernels. The researcher modified some vector-matrix barycentered Lagrange interpolation formulas in such a way that they are convenient for interpolating the kernel twice with respect to two kernel variables and new ideas for the choice of interpolation nodes are introduced to ensure the isolation of the singularity of the kernel of the interpolation nodes. Numerical results are presented in tables and figures.

Iranian researchers in turn offered a numerical method for solving Volterra and Fredholm integral equations of the second kind [9], which is based on the special representation of vector forms of triangular functions and the corresponding operational integration matrix, the integral equation is reduced to a linear system of algebraic equations. The results of numerical experiments are given.

A numerical method for solving linear and nonlinear Fredholm integral equations was proposed in [10]. In this method, the integral is approximated by Romberg's quadrature formula. Some numerical examples are given to investigate the applicability of the method.

The paper [11] is devoted to a new polynomial method for solving Volterra-Fredholm integral equations. The method is based on the use of shifted Lejandre polynomials together with the Gauss integration formula, by means of which the integral equations are reduced to a system of algebraic equations. Some examples are given to illustrate the method.

The researchers of [12] proposed a numerical method for solving linear Fredholm and Volterra integral equations of the second kind using Lejandre wavelets. The authors used a quadrature formula to calculate the products of any functions that are needed in the approximation of the integral equation. The problem is then reduced to solving a system of linear algebraic equation. The numerical results are given in tabular and graphical form.

Iraqi researchers in their work [13] presented a new technique for solving mixed Volterra-Fredholm integral equations of the second kind. The method is based on using a polynomial and transforming the integral equation to a linear programming problem to be solved by simplex method. To demonstrate the competence of the method and the accuracy of the results, a comparison between the exact and approximate solution is given by calculating the absolute error.

The authors of [14] developed a Nystrom method based on the anti-Gauss quadrature formula, which is investigated in terms of stability and convergence in suitable weighted spaces. The Nystrom interpolation corresponding to the Gauss and anti-Gauss quadrature formulas give upper and lower bounds for the solution of the equation under appropriate assumptions, which can be easily checked for a particular weight function. The accuracy of the solution can be improved by approximating it with an averaged Nystrom interpolation. The effectiveness of the method is illustrated by various examples.

The work of a Chinese researcher is devoted to the numerical solution of the Fredholm integral equation of the second kind [15], which is based on the application of the Gauss-Lobatto quadrature formula. The author gives the existence condition of the solution and error analysis. Numerical examples confirm the theoretical analysis and show the effectiveness of the considered algorithm.

A new method is proposed by the authors of [16] to solve the Fredholm integral equation using Hosoi polynomials derived from one standard class of graphs called paths. The researchers claim that the proposed algorithm extends the desired solution in terms of a set of continuous polynomials on the closed interval [0, 1]. The accuracy and efficiency depend on the set of Hosoi polynomials.

The authors of [17] used the Haar wavelet method for the numerical solution of one-dimensional and two-dimensional Fredholm integral equations of the second kind. The main idea of the Haar wavelet collocation method is to transform the integral equation into a system of algebraic equations. Numerical results in tabular and graphical forms are given.

Researchers in their work [18] applied the general Lejandre wavelet method to the Fredholm integral equation of the second kind. Using a general operational integration matrix, they approximated the Fredholm integral equation of the second kind by general Lejandre wavelets. Some examples are given to illustrate the method.

The main purpose of this paper is to show the application of the method of quadrature with derivatives for the numerical solution of the linear Fredholm integral equation of the second kind. In addition, to illustrate the behavior of the absolute error with the help of examples.

2 Method of weighted optimal quadrature formulas with derivatives

In [19], an optimal quadrature formula in the space $L_2^{(m)}(0,1)$ is given:

$$\int_0^1 p(s)\varphi(s)ds \cong \sum_{k=0}^{m-1}\sum_{\beta=0}^{N} C_\beta^{(k)}\varphi_\beta^{(k)}, \quad m=0,\ 1,.... \tag{1}$$

Here $p(s)$ is the weight function, $\varphi(x)$ is the given function $(\varphi_\beta = \varphi(h\beta)), C_\beta^{(k)}$ are the optimal coefficients of the quadrature formula, and h is the grid spacing.

The optimal coefficients of the quadrature formula (1) are determined using the following formulas:

$$C_0^{(k)} = \frac{p_k}{2} + \frac{(-1)^k}{h}[F_{k1} - F_{k0}], \tag{2}$$

$$C_\beta^{(k)} = \frac{(-1)^k}{h}[F_k(h(\beta-1)) - 2F_k(h\beta) + F_k(h(\beta+1))], \beta = 0,1,...,N, \tag{3}$$

$$C_N^{(k)} = \frac{p_k}{2} + \frac{(-1)^k}{h}[F_{kN-1} - F_{kN}]. \tag{4}$$

Here

$$F_{k\beta} = f_{k\beta} - \sum_{j=0}^{k-1}\sum_{\gamma=0}^{N}(-1)^j C_\gamma^{(j)}\frac{(h\beta-h\gamma)^{k-j+1}sign(h\beta-h\gamma)}{2(k-j+1)!}, \tag{5}$$

$$f_{k\beta} = (-1)^k \int_0^1 p(s)\frac{(s-h\beta)^{k+1} sign(s-h\beta)}{2(k+1)!}ds, \tag{6}$$

$$p_k = \frac{g_k}{k!} - \sum_{j=0}^{k-1}\sum_{\gamma=0}^{N} C_\gamma^{(j)} \frac{(h\gamma)^{k-j}}{(k-j)!}, \tag{7}$$

$$g_k = \int_0^1 p(s)s^k ds, \quad k = 0,\ 1, ..., m-1;\ m = 1,\ 2,\ .\ .\ .\ . \tag{8}$$

3 The main results

Consider the following linear Fredholm integral equation of the second kind:

$$y(x) - \lambda \int_a^b K(x,s)y(s)ds = f(x), x \in [a,b], \tag{9}$$

where $K(x,s)$ is the kernel of the integral equation, $f(x)$ is the free term (right-hand side) of the integral equation,λ is the parameter of the integral equation, a and b are the limits of integration, and $y(x)$ is the unknown function to be determined.

Applying the optimal quadrature formula (1) to the integral equation (9) and passing to the difference equations, we obtain:

$$y_i - \lambda \sum_{k=0}^{m-1}\sum_{\beta=0}^{N} C_{i\beta}^{(k)} y_\beta^{(k)} = f_i, \quad i = 0,1,...,N;\ m = 1,\ 2,\ .\ .\ .\ . \tag{10}$$

In (10), the number of equations is $N+1$, and the number of unknowns is $m(N+1)$. In order to obtain the remaining equations $(m-1)$ times we differentiate the integral equation (9) on the variable x:

$$\begin{gathered} y'(x) - \lambda \int_0^1 K'_x(x,s)y(s)ds = f'(x), \\ y''(x) - \lambda \int_0^1 K''_x(x,s)y(s)ds = f''(x), ..., \\ y^{(m-1)}(x) - \lambda \int_0^1 K_x^{(m-1)}(x,s)y(s)ds = f^{(m-1)}(x), \\ m = 0,1,.... \end{gathered} \tag{11}$$

Where

$$K^{(k)}(x,s) = \frac{d^k}{dx^k}K(x,s), f^{(k)}(x) = \frac{d^k}{dx^k}f(x),$$
$$k = 0,1,...,m = 1,2,....$$

Then applying the optimal quadrature formula (1) for each equation (11) and passing to the difference equations, we obtain a system of linear algebraic equations to determine the values of the desired function $y(x_i)$ and the values of the derivatives $y'(x_i), y''(x_i), ..., \quad y^{(m-1)}(x_i)$ at the grid nodes $x_i (i = 0,1,...,N)$:

$$y_i - \lambda \sum_{k=0}^{m-1}\sum_{\beta=0}^{N} C_{i\beta}^{(k)} y_\beta^{(k)} = f_i,$$

$$y_i' - \lambda \sum_{k=0}^{m-1} \sum_{\beta=0}^{N} C_{i\beta}^{(k)} y_\beta^{(k)} = f_i', ..., \tag{12}$$

$$y_i^{(m-1)} - \lambda \sum_{k=0}^{m-1} \sum_{\beta=0}^{N} C_{i\beta}^{(k)} y_\beta^{(k)} = f_i^{(m-1)},$$

$$i = 0, 1, ..., N;\ m = 1,\ 2,\ ....$$

Then to calculate the coefficients of the system of algebraic equations (12) in the place of formulas (2) - (8) we obtain:

$$C_{i0}^{(k)} = \frac{p_{ik}}{2} + \frac{(-1)^k}{h}[F_{ik1} - F_{ik0}], \tag{13}$$

$$C_{i\beta}^{(k)} = \frac{(-1)^k}{h}[F_{ik\beta-1} - 2F_{ik\beta} + F_{ik\beta+1}], \tag{14}$$

$$C_{iN}^{(k)} = \frac{p_{ik}}{2} + \frac{(-1)^k}{h}[F_{ikN-1} - F_{ikN}], \tag{15}$$

$$F_{ik\beta} = f_{ik\beta} - \sum_{j=0}^{k-1} \sum_{\gamma=0}^{N} (-1)^j C_{i\gamma}^{(j)} \frac{(h\beta - h\gamma)^{k-j+1} sign(h\beta - h\gamma)}{2(k-j+1)!}, \tag{16}$$

$$f_{ik\beta} = (-1)^k \int_0^1 K^{(k)}(x_i, s) \frac{(s - h\beta)^{k+1} sign(s - h\beta)}{2(k+1)!} ds, \tag{17}$$

$$p_{ik} = \frac{g_{ik}}{k!} - \sum_{j=0}^{k-1} \sum_{\gamma=0}^{N} C_{i\gamma}^{(j)} \frac{(h\gamma)^{k-j}}{(k-j)!}, \tag{18}$$

$$g_{ik} = \int_0^1 K^{(k)}(x_i, s) s^k ds, \quad k = 0,\ 1, ..., m-1;\ m = 1,\ 2,\ .\ .\ .\ , \tag{19}$$

$$K^{(k)}(x_i, s) = \frac{d^k}{dx^k}\ K(x_i, s), \quad f_i^{(k)} = \frac{d^k}{dx^k} f(x_i),$$

$$\beta = 0,\ 1, ...,\ N,\ i = 0,\ 1,\ ..., N,\ k = 0,\ 1,\ ..., m-1;\ m = 1,\ 2,\ ....$$

Now let us turn to concrete examples for approbation of the above algorithm. For this purpose, let us take a family of linear Fredholm integral equations of the second kind [7]:

$$y(x) - \lambda \int_0^1 G(x) H(s) y(s) ds = f(x), x \in [0, 1]. \tag{20}$$

Here the kernel of the integral equation is $K(x, s) = G(x)H(s)$. In (20), the function $G(x)$ can be taken before the integral. Then (20) and (11) will have the following form:

$$y(x) - \lambda G(x) \int_0^1 H(s) y(s) ds = f(x), \quad x \in [0, 1],$$

$$y'(x) - \lambda G'(x) \int_0^1 H(s) y(s) ds = f'(x), ..., \tag{21}$$

$$y^{(m-1)}(x) - \lambda G^{(m-1)}(x) \int_0^1 H\,(s)\, y(s) ds = f^{(m-1)}(x),$$

$$m = 1,\ 2,\ ....$$

Replacing the integrals in (21) by the quadrature formula (1), we obtain the following system of linear algebraic equations:

$$y_i - \lambda G_i \sum_{k=0}^{m-1} \sum_{\beta=0}^{N} C_\beta^{(k)} y_\beta^{(k)} = f_i,$$

$$y_i' - \lambda G_i' \sum_{k=0}^{m-1} \sum_{\beta=0}^{N} C_\beta^{(k)} y_\beta^{(k)} = f_i', ..., \tag{22}$$

$$y_i^{(m-1)} - \lambda G_i^{(m-1)} \sum_{k=0}^{m-1} \sum_{\beta=0}^{N} C_\beta^{(k)} y_\beta^{(k)} = f_i^{(m-1)},$$

$$i = 0, 1, \ldots, N;\ m = 1,\ 2,\ \ldots.$$

Where

$$G_i^{(k)} = \frac{d^k}{dx^k} G(x_i), i = 0,\ 1,\ ..., N,\quad k = 0,\ 1,\ ..., m-1,\ m = 1,\ 2,\ ....$$

To calculate $C_\beta^{(k)}$-coefficients of the optimal quadrature formula, formulas (2) through (8) are used. In (6) and (8), $H(s)$ is used instead of $p(s)$:

$$f_{k\beta} = (-1)^k \int_0^1 H(s) \frac{(s-h\beta)^{k+1} sign(s-h\beta)}{2(k+1)!} ds,$$

$$g_k = \int_0^1 H(s) s^k ds,\quad k = 0,\ 1, ..., m-1;\ m = 1,\ 2,\ \ldots\ .$$

If (22) is represented in matrix form

$$AY\ =\ B, \tag{23}$$

then the elements of the matrix A are determined by the following formulas:

$$A_{00} = 1 - \lambda G_0^{(0)} C_0^{(0)}, A_{01} = -\lambda G_0^{(0)} C_1^{(0)}, ..., A_{0N} = -\lambda G_0^{(0)} C_N^{(0)},$$

$$A_{0N+1} = -\lambda G_0^{(0)} C_0^{(1)},\quad A_{0N+2} = -\lambda G_0^{(0)} C_1^{(1)},\ \ldots,\ A_{02N+1} = -\lambda G_0^{(0)} C_N^{(1)},\ \ldots,$$

$$A_{0k(N+1)} = -\lambda G_0^{(0)} C_0^{(k)},\ \ldots,\quad A_{0(k+1)N+k} = -\lambda G_0^{(0)} C_N^{(k)},\ \ldots,$$

$$A_{0(m-1)(N+1)} = -\lambda G_0^{(0)} C_0^{(m-1)},\ \ldots,\ A_{0M} = -\lambda G_0^{(0)} C_N^{(k)},$$

$$A_{10} = -\lambda G_1^{(0)} C_0^{(0)},\ A_{11} = 1 - \lambda G_1^{(0)} C_1^{(0)}, ...,\ A_{1N} = -\lambda G_1^{(0)} C_N^{(0)},$$

$$A_{1N+1} = -\lambda G_1^{(0)} C_0^{(1)},\ \ A_{1N+2} = -\lambda G_1^{(0)} C_1^{(1)},\ \ldots,\ A_{12N+1} = -\lambda G_1^{(0)} C_N^{(1)},\ \ldots,$$

$$A_{1k(N+1)} = -\lambda G_1^{(0)} C_0^{(k)},\ \ldots,\quad A_{1(k+1)N+k} = -\lambda G_1^{(0)} C_N^{(k)},\ \ldots,$$

$$A_{1(m-1)(N+1)} = -\lambda G_1^{(0)} C_0^{(m-1)},\ \ldots,\ A_{1M} = -\lambda G_1^{(0)} C_N^{(k)}, ...,$$

$$A_{N0} = -\lambda G_N^{(0)} C_0^{(0)}, A_{N1} = -\lambda G_N^{(0)} C_1^{(0)}, ...,\ A_{NN} = 1 - \lambda G_1^{(0)} C_N^{(0)},$$

$$A_{1N+1} = -\lambda G_1^{(0)} C_0^{(1)},\ A_{1N+2} = -\lambda G_1^{(0)} C_1^{(1)},\ \ldots,\ A_{12N+1} = -\lambda G_1^{(0)} C_N^{(1)},\ \ldots,$$

$$A_{1k(N+1)}=-\lambda G_1^{(0)}C_0^{(k)},\ \ldots,\ A_{1(k+1)N+k}=-\lambda G_1^{(0)}C_N^{(k)},\ \ldots,$$

$$A_{1(m-1)(N+1)}=-\lambda G_1^{(0)}C_0^{(m-1)},\ \ldots,\ A_{1M}=-\lambda G_1^{(0)}C_N^{(k)},\ldots,$$

$$A_{l0}=-\lambda G_0^{(k)}C_0^{(0)},\ A_{l1}=-\lambda G_0^{(k)}C_1^{(0)},\ \ldots,\ A_{lN}=-\lambda G_0^{(k)}C_N^{(0)},$$

$$A_{lN+1}=-\lambda G_0^{(k)}C_0^{(1)},\ \ A_{lN+2}=-\lambda G_0^{(k)}C_1^{(1)},\ \ldots,\ A_{l2N+1}=-\lambda G_0^{(k)}C_N^{(1)},\ \ldots,$$

$$A_{ll}=1-\lambda G_0^{(k)}C_0^{(k)},\ \ldots,\ A_{l(k+1)N+k}=-\lambda G_0^{(k)}C_N^{(k)},\ \ldots,$$

$$A_{l(m-1)(N+1)}=-\lambda G_0^{(k)}C_0^{(m-1)},\ \ldots,\ A_{lM}=-\lambda G_0^{(k)}C_N^{(k)},\ldots,$$

$$A_{\alpha 0}=-\lambda G_\alpha^{(k)}C_0^{(0)},\ A_{\alpha 1}=-\lambda G_\alpha^{(k)}C_1^{(0)},\ldots,\ A_{\alpha N}=-\lambda G_\alpha^{(k)}C_N^{(0)},$$

$$A_{\alpha N+1}=-\lambda G_\alpha^{(k)}C_0^{(1)},\ \ A_{\alpha N+2}=-\lambda G_\alpha^{(k)}C_1^{(1)},\ \ldots,\ A_{\alpha 2N+1}=-\lambda G_\alpha^{(k)}C_N^{(1)},\ \ldots,$$

$$A_{\alpha k(N+1)}=-\lambda G_\alpha^{(k)}C_0^{(k)},\ \ldots,\ A_{\alpha(k+1)N+k}=1-\lambda G_\alpha^{(k)}C_N^{(k)},\ \ldots,$$

$$A_{\alpha(m-1)(N+1)}=-\lambda G_\alpha^{(k)}C_0^{(m-1)},\ \ldots,\ A_{\alpha M}=-\lambda G_\alpha^{(k)}C_N^{(k)},\ldots,$$

$$A_{\gamma 0}=-\lambda G_0^{(m-1)}C_0^{(0)},\ A_{\gamma 1}=-\lambda G_0^{(m-1)}C_1^{(0)},\ \ldots,\ A_{\gamma N}=-\lambda G_0^{(m-1)}C_N^{(0)},$$

$$A_{\gamma N+1}=-\lambda G_0^{(m-1)}C_0^{(1)},\ \ A_{\gamma N+2}=-\lambda G_0^{(m-1)}C_1^{(1)},\ \ldots,\ A_{\gamma 2N+1}=-\lambda G_0^{(m-1)}C_N^{(1)},\ldots,$$

$$A_{\gamma\gamma}=1-\lambda G_0^{(m-1)}C_0^{(k)},\ \ldots,\ A_{\gamma(k+1)N+k}=-\lambda G_0^{(m-1)}C_N^{(k)},\ \ldots,$$

$$A_{\gamma(m-1)(N+1)}=-\lambda G_0^{(m-1)}C_0^{(m-1)},\ \ldots,\ A_{\gamma M}=-\lambda G_0^{(m-1)}C_N^{(k)},\ldots,$$

$$A_{M0}=-\lambda G_M^{(m-1)}C_0^{(0)},A_{M1}=-\lambda G_M^{(m-1)}C_1^{(0)},\ldots,\ A_{MN}=-\lambda G_M^{(m-1)}C_N^{(0)},$$

$$A_{MN+1}=-\lambda G_M^{(m-1)}C_0^{(1)},\ A_{MN+2}=-\lambda G_M^{(m-1)}C_1^{(1)},\ldots,A_{M2N+1}=-\lambda G_M^{(m-1)}C_N^{(1)},\ldots,$$

$$A_{Mk(N+1)}=-\lambda G_M^{(m-1)}C_0^{(k)},\ \ldots,\ A_{M(k+1)N+k}=-\lambda G_M^{(m-1)}C_N^{(k)},\ \ldots,$$

$$A_{M(m-1)(N+1)}=-\lambda G_M^{(m-1)}C_0^{(m-1)},\ \ldots,\ A_{MM}=1-\lambda G_M^{(m-1)}C_N^{(k)},$$

The elements of the free term B are defined as follows

$$B_1=f_1^{(0)},\ B_2=f_2^{(0)},\ \ldots,\ B_N=f_N^{(0)},$$

$$B_{N+1}=f_{N+1}^{(1)},\ B_{N+2}=f_2^{(1)},\ \ldots,\ B_{2N+1}=f_{2N+1}^{(1)},\ldots,$$

$$B_\gamma=f_\gamma^{(m-1)},\ B_{\gamma+1}=f_{\gamma+1}^{(m-1)},\ \ldots,\ B_M=f_M^{(m-1)}.$$

The vector of the sought function Y has the following form:

$$Y_1=y_1^{(0)},\ Y_2=y_2^{(0)},\ \ldots,\ Y_N=y_N^{(0)},$$

$$Y_{N+1}=y_{N+1}^{(1)},\ Y_{N+2}=y_2^{(1)},\ \ldots,\ Y_{2N+1}=y_{2N+1}^{(1)},\ldots,$$

$$Y_\gamma=y_\gamma^{(m-1)},\ Y_{\gamma+1}=y_{\gamma+1}^{(m-1)},\ \ldots,\ Y_M=y_M^{(m-1)},$$

where

$$y_i^{(0)}=y_i,\ G_i^{(0)}=G_i,\ f_i^{(0)}=f_i,\quad i=0,\ 1,\ \ldots,\ N,$$

$$M=m(N+1)-1,\ l=k(N+1),\ \alpha=l+N,\ \gamma=M-N.$$

Solving the system (23) we define the vector Y. The first $N+1$ elements of this vector are the values of the required function of the integral equation (20), i.e.

$$y_i = Y_i, \quad i = 0,\ 1,\quad .\ .\ .\ .\ ,\ N.$$

4 Approbation of the method

A program in Maple language is compiled according to this algorithm. In addition to numerical results, the program provides formulas for calculating the coefficients of the optimal quadrature formula.

In the following examples it is necessary to solve the integral equations by the quadrature method using the optimal quadrature formula at m=1, 2, 3, 4. Compare the results with the exact solution and analyze the errors. The exact solutions of the examples are determined using the formulas given in [20].

Example 1. In (13), $K(x,s) = e^{-s}$, $f(x) = e^x - 1$, $\lambda = 1$. Then the integral equation (13) will take the following form:

$$y(x) - \lambda \int_0^1 e^{-s} y(s) ds = e^{-x}, x \in [0;1]. \tag{16}$$

The exact solution of the integral equation (16) is:

$$y(x) = e^x.$$

The formulas for calculating the coefficients of the optimal quadrature formula are given below:

$$C_0^{(0)} = (e^h - h - 1)/h,\ C_N^{(0)} = (e^{(N-1)h} - e^{Nh} he)/h,$$

$$C_\beta^{(0)} = e^{h(\beta-1)}(e^h - 1)^2/h, \quad \beta = 1,\ 2,\ ..., N-1,$$

$$C_0^{(1)} = (e^h(h-2) + h + he)/(2h),$$

$$C_\beta^{(1)} = e^{h(\beta-1)}(e^h - 1)(e^h(h-2) + h + he)/(2h),\ \beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(1)} = ((2+h)(e^{Nh} - e^{(N-1)h}) - 2Nh^2 e)/(2h),$$

$$C_0^{(2)} = (e^h(12 - 6h + h^2) - 6h - h^2 - 12)/(12h),$$

$$C_\beta^{(2)} = e^{h(\beta-1)}\left(e^h - 1\right)\left(e^h(12 - 6h + h^2) - 6h - h^2 - 12\right)/\left(2h\right),$$

$$\beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(2)} = (e^{(N-1)h}(12 + h^2 + 6h) - e^{Nh}(12-$$

$$-6h) + e(6N^2 + 6h))/(12h),$$

$$C_0^{(3)} = -(e^h(12 - 6h + h^2) - 6h - h^2 - 12)/(12h),$$

$$C_\beta^{(3)} = -e^{h(\beta-1)}\left(e^h - 1\right)\left(e^h(12 - 6h + h^2) - 6h - h^2 - 12\right)/\left(2h\right),$$

$$\beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(3)} = -(e^{(N-1)h}(12 + h^2 + +6h) - e^{Nh}(12+$$

$$+6h + h^2) + e(6Nh^2 - 2N^2h^4 + 4h))/(12h).$$

The results for this example are shown in Table 1a,b,c and Figure 1a,b,c for $h=0.1$, 0.05, 0.025.

Table 1a, Results of solving the integral equation (Example 1, K(x,s)=exp(-s), f(x)=exp(x)-1, y(x)=exp(x), h=0.1)

x_j	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	1,0022660	0,9999992	0,9999996	1,0000000	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,10	1,1051709	1,1074369	1,1051702	1,1051705	1,1051709	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,20	1,2214028	1,2236687	1,2214020	1,2214024	1,2214028	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,30	1,3498588	1,3521248	1,3498581	1,3498584	1,3498588	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,40	1,4918247	1,4940907	1,4918239	1,4918243	1,4918247	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,50	1,6487213	1,6509873	1,6487205	1,6487209	1,6487213	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,60	1,8221188	1,8243848	1,8221180	1,8221184	1,8221188	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,70	2,0137527	2,0160187	2,0137520	2,0137523	2,0137527	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,80	2,2255409	2,2278069	2,2255402	2,2255406	2,2255409	2,27E-03	7,55E-07	3,78E-07	1,80E-10
0,90	2,4596031	2,4618691	2,4596024	2,4596027	2,4596031	2,27E-03	7,55E-07	3,78E-07	1,80E-10
1,00	2,7182818	2,7205478	2,7182811	2,7182815	2,7182818	2,27E-03	7,55E-07	3,78E-07	1,80E-10
					Maximum absolute error	**2,27E-03**	**7,55E-07**	**3,78E-07**	**1,80E-10**

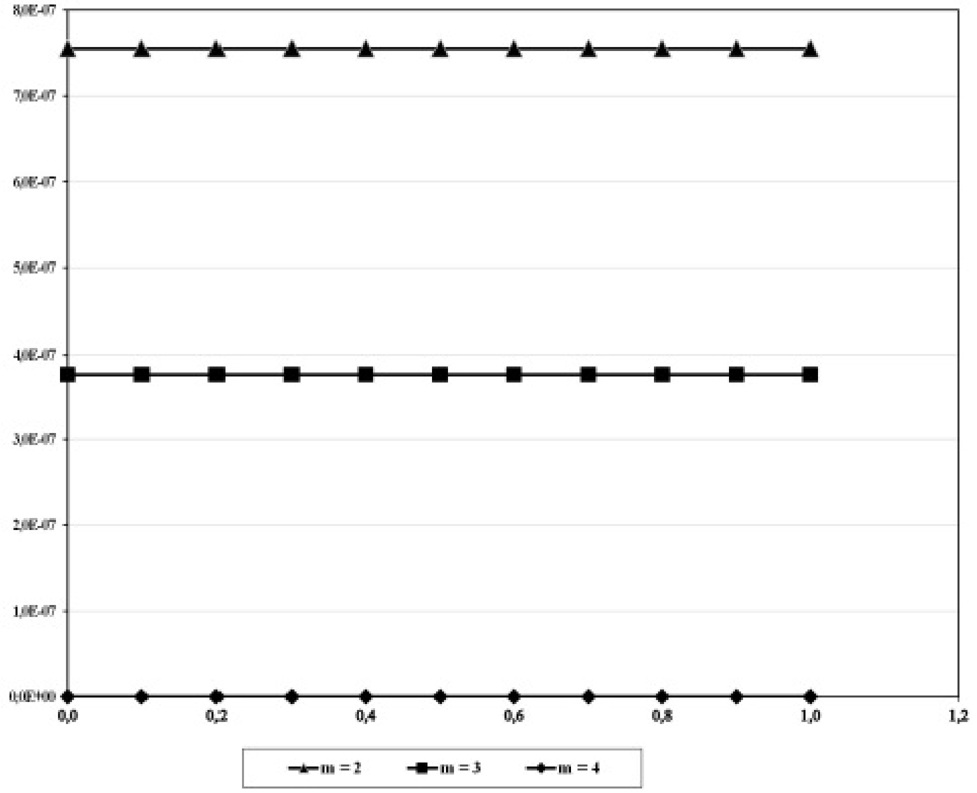

Figure 1a. Graphical illustration of the absolute error (Example 1, K(x,s)=exp(-s), f(x)=exp(x)-1, y(x)=exp(x), h=0.1)

Table 1a, Results of solving the integral equation (Example 1, K(x,s)=exp(-s), f(x)=exp(x)-1, y(x)=exp(x), h=0.05)

x_j	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	1,0005664	1,0000000	1,0000000	1,0000000	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,10	1,1051709	1,1057373	1,1051709	1,1051709	1,1051709	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,20	1,2214028	1,2219691	1,2214027	1,2214027	1,2214028	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,30	1,3498588	1,3504252	1,3498588	1,3498588	1,3498588	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,40	1,4918247	1,4923911	1,4918247	1,4918247	1,4918247	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,50	1,6487213	1,6492876	1,6487212	1,6487212	1,6487213	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,60	1,8221188	1,8226852	1,8221188	1,8221188	1,8221188	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,70	2,0137527	2,0143191	2,0137527	2,0137527	2,0137527	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,80	2,2255409	2,2261073	2,2255409	2,2255409	2,2255409	5,66E-04	4,72E-08	2,36E-08	2,81E-12
0,90	2,4596031	2,4601695	2,4596031	2,4596031	2,4596031	5,66E-04	4,72E-08	2,36E-08	2,81E-12
1,00	2,7182818	2,7188482	2,7182818	2,7182818	2,7182818	5,66E-04	4,72E-08	2,36E-08	2,81E-12
					Maximum absolute error	**5,66E-04**	**4,72E-08**	**2,36E-08**	**2,81E-12**

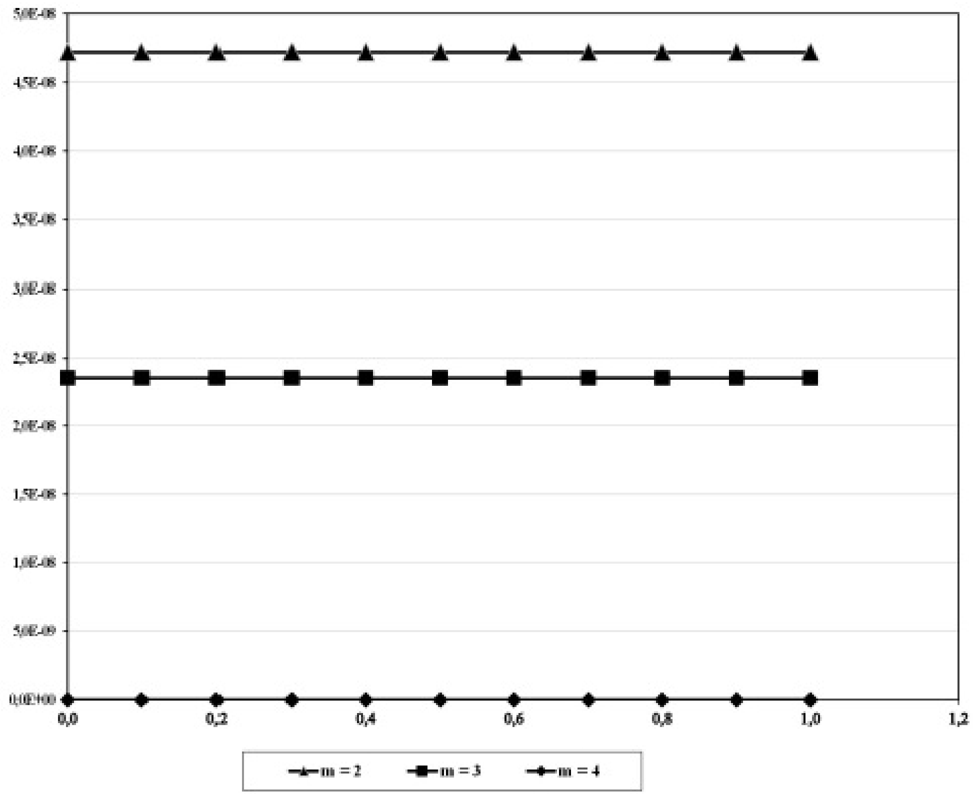

Figure 1a. Graphical illustration of the absolute error (Example 1, K(x,s)=exp(-s), f(x)=exp(x)-1, y(x)=exp(x), h=0.05)

Table 1a, Results of solving the integral equation (Example 1, K(x,s)=exp(-s), f(x)=exp(x)-1, y(x)=exp(x), h=0.025)

x_j	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	1,0001416	1,0000000	1,0000000	1,0000000	1,42E-04	2,95E-09	1,47E-09	4,49E-14
0,10	1,1051709	1,1053125	1,1051709	1,1051709	1,1051709	1,42E-04	2,95E-09	1,47E-09	4,60E-14
0,20	1,2214028	1,2215443	1,2214028	1,2214028	1,2214028	1,42E-04	2,95E-09	1,47E-09	4,62E-14
0,30	1,3498588	1,3500004	1,3498588	1,3498588	1,3498588	1,42E-04	2,95E-09	1,47E-09	4,60E-14
0,40	1,4918247	1,4919663	1,4918247	1,4918247	1,4918247	1,42E-04	2,95E-09	1,47E-09	4,64E-14
0,50	1,6487213	1,6488629	1,6487213	1,6487213	1,6487213	1,42E-04	2,95E-09	1,47E-09	4,66E-14
0,60	1,8221188	1,8222604	1,8221188	1,8221188	1,8221188	1,42E-04	2,95E-09	1,47E-09	4,66E-14
0,70	2,0137527	2,0138943	2,0137527	2,0137527	2,0137527	1,42E-04	2,95E-09	1,47E-09	4,40E-14
0,80	2,2255409	2,2256825	2,2255409	2,2255409	2,2255409	1,42E-04	2,95E-09	1,47E-09	4,40E-14
0,90	2,4596031	2,4597447	2,4596031	2,4596031	2,4596031	1,42E-04	2,95E-09	1,47E-09	4,44E-14
1,00	2,7182818	2,7184234	2,7182818	2,7182818	2,7182818	1,42E-04	2,95E-09	1,47E-09	4,44E-14
					Maximum absolute error	**1,42E-04**	**2,95E-09**	**1,47E-09**	**4,66E-14**

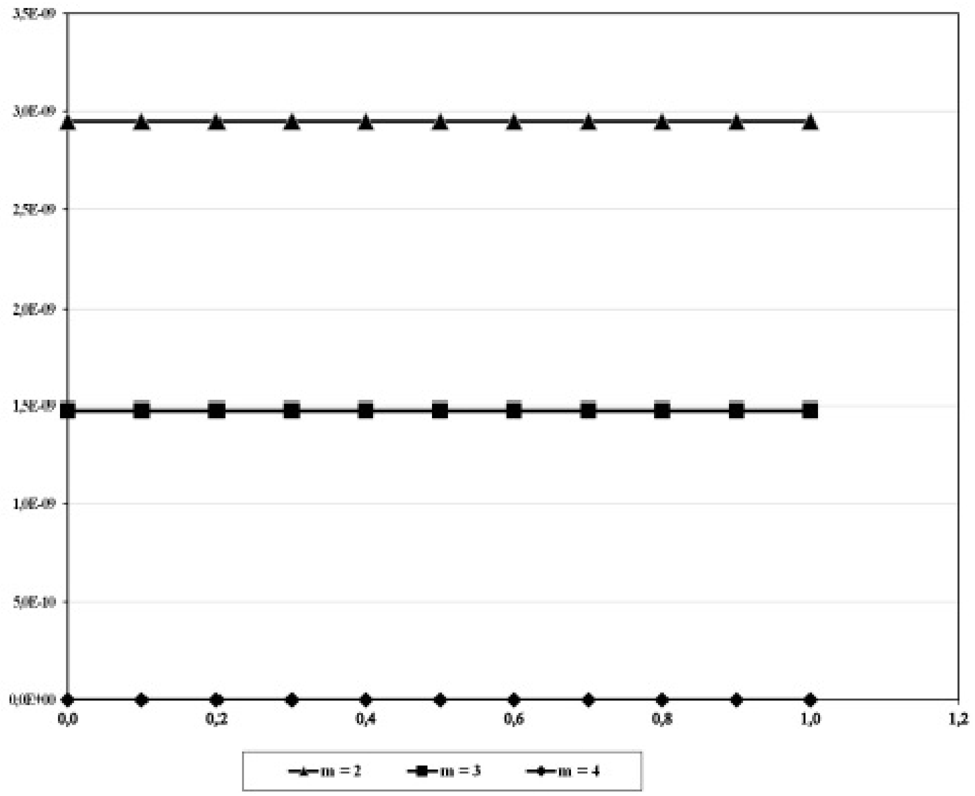

Figure 1a. Graphical illustration of the absolute error (Example 1, K(x,s)=exp(-s), f(x)=exp(x)-1, y(x)=exp(x), h=0.025)

Example 2. In (13) $K(x,s) = e^(2x - 5s/3)$, $f(x) = e^{2x+1/3}$, $\lambda = 1/3$.Then the integral equation (13) will take the following form:

$$y(x) + \frac{1}{3}\int_0^1 e^{2x-5s/3}y(s)ds = e^{2x+1/3}, x \in [0;1]. \tag{17}$$

The exact solution of the integral equation (17) is:

$$y(x) = e^{2x}.$$

The formulas for calculating the coefficients of the optimal quadrature formula are given below:

$$C_0^{(0)} = (e^h(h-2) + h + 2)/h,$$

$$C_\beta^{(0)} = e^{h(\beta-1)}\left(e^h - 1\right)\left((2-(\beta+1)h)e^h + (\beta-1)h - 2\right)/h,$$

$$\beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(0)} = e^{(N-1)h}((N-1)h - 2 + (2 - Nh)e^h)/h,$$

$$C_0^{(1)} = ((\ h^2 - 4h + 6)e^h - 2h - 6)/(2h),$$

$$C_\beta^{(1)} = -e^{h(\beta-1)}((-(\beta+1)\ h^2 + 2(\beta+2)h - 6)e^{2h} + (12 - 4\beta h)e^h +$$

$$+(\beta-1)\ h^2 + 2(\beta-2)h - 6)/(2h),$$

$$\beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(1)} = ((6 - 2(N-2)h - (N-1)\ h^2)e^{(N-1)h} + (N\ h^2 +$$

$$+2(N-1)h - 6)e^{Nh} + 2he)/(2h),$$

$$C_0^{(2)} = (e^h(h^3 - 8h^2 + 30h - 48) + 2h^2 + 18h + 48)/(12h);$$

$$C_\beta^{(2)} = e^{h(\beta-1)}(e^{2h}((\beta+1)h^3 - (6\beta+8)h^2 + (12\beta+30)h - 48) +$$

$$+e^h(-2\beta h^3 + 4h^2 - 24\beta h + 96) + (\beta-1)h^3 +$$

$$+(6\beta+8)h^2 + (12\beta-30)h - 48)/\left(2h\right),$$

$$\beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(2)} = (e^{(N-1)h}((N-1)h^3 + (6N-8)h^2 + (12N-30)h - 48) +$$

$$+e^{Nh}(-Nh^3 - (6N-2)h^2 - (12N-18)h + 48) - 12he(Nh-1))/(12h),$$

$$C_0^{(3)} = (e^h(-h^3 + 9h2 - 36h + 60) - 3h^2 - 24h - 60)/(12h),$$

$$C_\beta^{(3)} = e^{h(\beta-1)}\left(e^{2h}(-(\beta+1)h^3 + (6\beta+9)h^2 - (12\beta+36)h + 60\right) +$$

$$+e^h(2\beta h^3 - 6h^2 + 24\beta h - 120) - (\beta-1)h^3 + (9-6\beta)h^2 -$$

$$-12h(\beta-3) + 60)/(12h),$$

$$\beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(3)} = (e^{(N-1)h}(-5h^3 - 27h^2 - 36h + 60) + e^{Nh}(6h^3 + 33h^2 +$$

$$+48h - 60) + he(216h + 72h + 18))/(12h).$$

The results for this example are shown in Table 2a,b,c and Figure 2a,b,c for h= 0.1, 0.05, 0.025.

Table 2a, Results of solving the integral equation (Example 2, K(x,s)=-exp(2*x-5*s/3)/3, f(x)=exp(2*x+1/3), y(x)=exp(2*x), h=0.1)

x_j	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	0,9990550	1,0000012	1,0000006	1,0000000	9,45E-04	1,16E-06	6,30E-07	1,10E-09
0,10	1,2214028	1,2202485	1,2214042	1,2214035	1,2214028	1,15E-03	1,41E-06	7,70E-07	1,34E-09
0,20	1,4918247	1,4904149	1,4918264	1,4918256	1,4918247	1,41E-03	1,72E-06	9,40E-07	1,64E-09
0,30	1,8221188	1,8203968	1,8221209	1,8221199	1,8221188	1,72E-03	2,11E-06	1,15E-06	2,01E-09
0,40	2,2255409	2,2234377	2,2255435	2,2255423	2,2255409	2,10E-03	2,57E-06	1,40E-06	2,45E-09
0,50	2,7182818	2,7157130	2,7182850	2,7182835	2,7182818	2,57E-03	3,14E-06	1,71E-06	2,99E-09
0,60	3,3201169	3,3169793	3,3201208	3,3201190	3,3201169	3,14E-03	3,84E-06	2,09E-06	3,66E-09
0,70	4,0552000	4,0513677	4,0552047	4,0552025	4,0552000	3,83E-03	4,69E-06	2,56E-06	4,47E-09
0,80	4,9530324	4,9483516	4,9530382	4,9530355	4,9530324	4,68E-03	5,73E-06	3,12E-06	5,45E-09
0,90	6,0496475	6,0439303	6,0496545	6,0496513	6,0496475	5,72E-03	6,99E-06	3,81E-06	6,66E-09
1,00	7,3890561	7,3820732	7,3890646	7,3890608	7,3890561	6,98E-03	8,54E-06	4,66E-06	8,14E-09
					Maximum absolute error	**6,98E-03**	**8,54E-06**	**4,66E-06**	**8,14E-09**

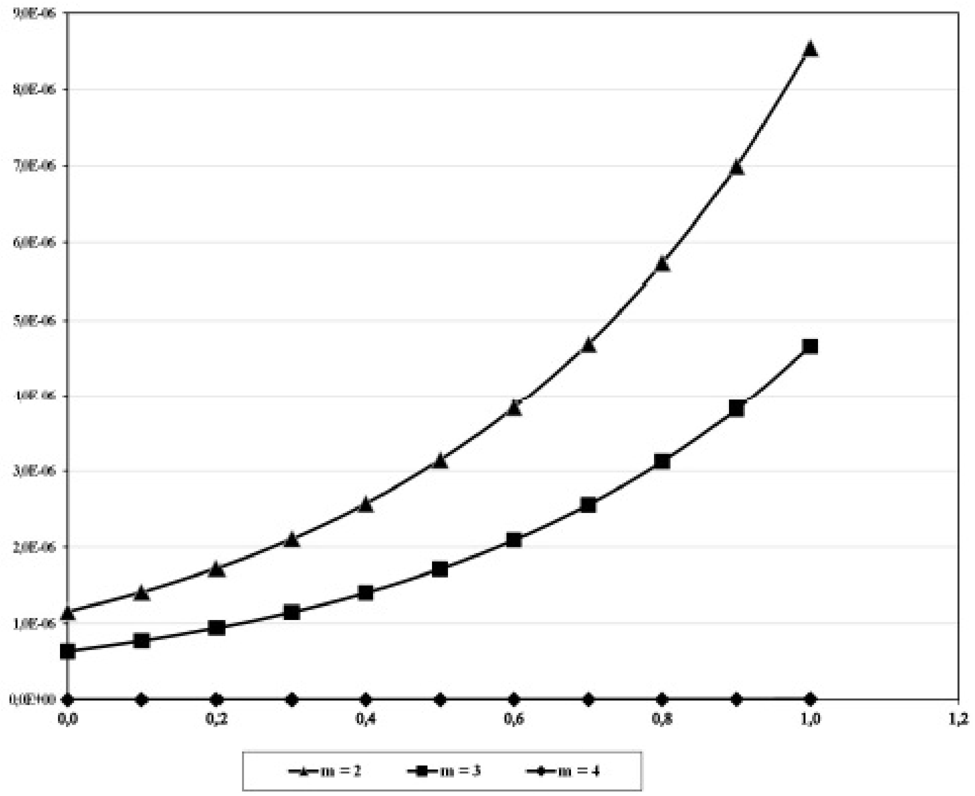

Figure 2a. Graphical illustration of the absolute error (Example 2, K(x,s)=-exp(2*x-5*s/3)/3, f(x)=exp(2*x+1/3), y(x)=exp(2*x), h=0.1)

Table 2b, Results of solving the integral equation (Example 2, K(x,s)=-exp(2*x-5*s/3)/3, f(x)=exp(2*x+1/3), y(x)=exp(2*x), h=0.05)

x_i	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	0,9997638	1,0000001	1,0000000	1,0000000	2,36E-04	7,22E-08	3,94E-08	1,72E-11
0,10	1,2214028	1,2211142	1,2214028	1,2214028	1,2214028	2,89E-04	8,82E-08	4,81E-08	2,10E-11
0,20	1,4918247	1,4914723	1,4918248	1,4918248	1,4918247	3,52E-04	1,08E-07	5,87E-08	2,56E-11
0,30	1,8221188	1,8216884	1,8221189	1,8221189	1,8221188	4,30E-04	1,32E-07	7,17E-08	3,13E-11
0,40	2,2255409	2,2250152	2,2255411	2,2255410	2,2255409	5,26E-04	1,61E-07	8,76E-08	3,83E-11
0,50	2,7182818	2,7176397	2,7182820	2,7182819	2,7182818	6,42E-04	1,96E-07	1,07E-07	4,67E-11
0,60	3,3201169	3,3193326	3,3201172	3,3201171	3,3201169	7,84E-04	2,40E-07	1,31E-07	5,71E-11
0,70	4,0552000	4,0542420	4,0552003	4,0552001	4,0552000	9,58E-04	2,93E-07	1,60E-07	6,97E-11
0,80	4,9530324	4,9518624	4,9530328	4,9530326	4,9530324	1,17E-03	3,58E-07	1,95E-07	8,51E-11
0,90	6,0496475	6,0482183	6,0496479	6,0496477	6,0496475	1,43E-03	4,37E-07	2,38E-07	1,04E-10
1,00	7,3890561	7,3873106	7,3890566	7,3890564	7,3890561	1,75E-03	5,33E-07	2,91E-07	1,27E-10
					Maximum absolute error	**1,75E-03**	**5,33E-07**	**2,91E-07**	**1,27E-10**

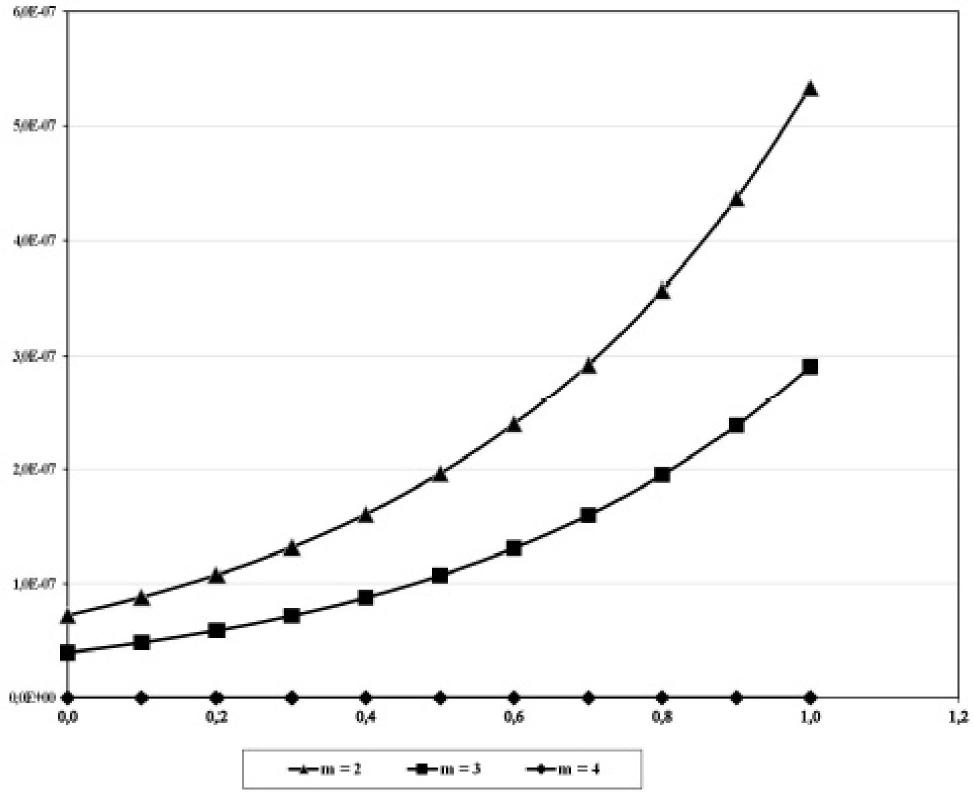

Figure 2b. Graphical illustration of the absolute error (Example 2, K(x,s)=-exp(2*x-5*s/3)/3, f(x)=exp(2*x+1/3), y(x)=exp(2*x), h=0.05)

Table 2c, Results of solving the integral equation (Example 2, K(x,s)=-exp(2*x-5*s/3)/3, f(x)=exp(2*x+1/3), y(x)=exp(2*x), h=0.025)

x_j	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	0,9999409	1,0000000	1,0000000	1,0000000	5,91E-05	4,51E-09	2,46E-09	2,68E-13
0,10	1,2214028	1,2213306	1,2214028	1,2214028	1,2214028	7,21E-05	5,51E-09	3,01E-09	3,29E-13
0,20	1,4918247	1,4917366	1,4918247	1,4918247	1,4918247	8,81E-05	6,73E-09	3,67E-09	4,00E-13
0,30	1,8221188	1,8220112	1,8221188	1,8221188	1,8221188	1,08E-04	8,22E-09	4,48E-09	4,88E-13
0,40	2,2255409	2,2254095	2,2255409	2,2255409	2,2255409	1,31E-04	1,00E-08	5,48E-09	5,99E-13
0,50	2,7182818	2,7181213	2,7182818	2,7182818	2,7182818	1,61E-04	1,23E-08	6,69E-09	7,31E-13
0,60	3,3201169	3,3199208	3,3201169	3,3201169	3,3201169	1,96E-04	1,50E-08	8,17E-09	8,91E-13
0,70	4,0552000	4,0549605	4,0552000	4,0552000	4,0552000	2,39E-04	1,83E-08	9,98E-09	1,09E-12
0,80	4,9530324	4,9527399	4,9530324	4,9530324	4,9530324	2,93E-04	2,23E-08	1,22E-08	1,33E-12
0,90	6,0496475	6,0492902	6,0496475	6,0496475	6,0496475	3,57E-04	2,73E-08	1,49E-08	1,63E-12
1,00	7,3890561	7,3886197	7,3890561	7,3890561	7,3890561	4,36E-04	3,33E-08	1,82E-08	1,98E-12
					Maximum absolute error	4,36E-04	3,33E-08	1,82E-08	1,98E-12

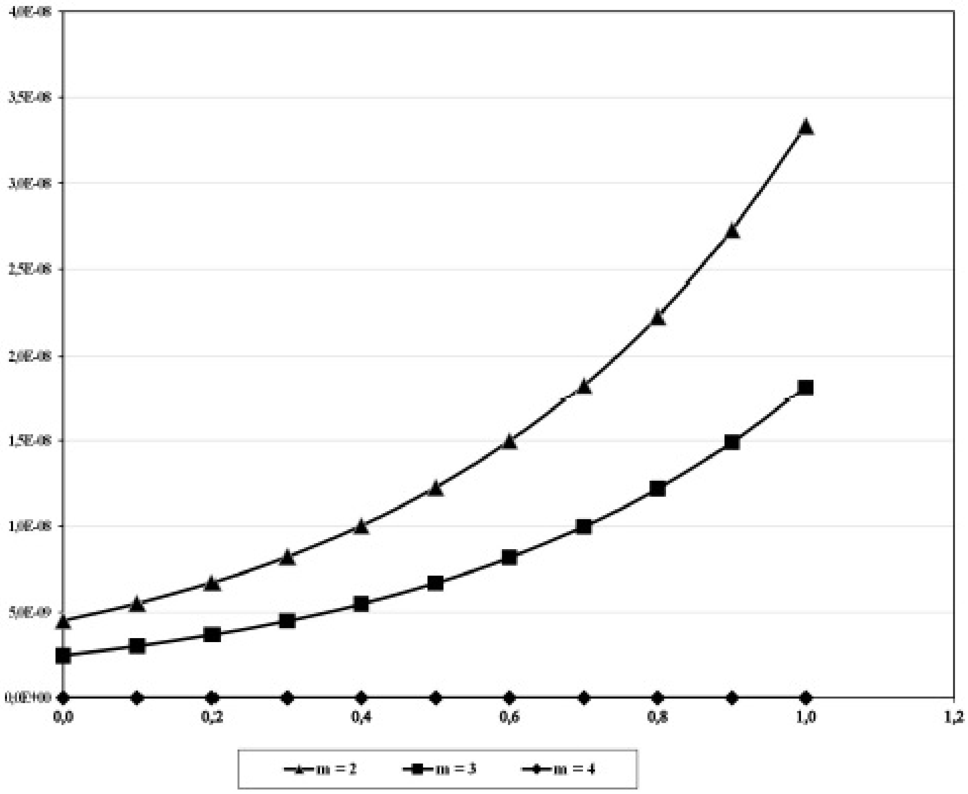

Figure 2c. Graphical illustration of the absolute error (Example 2, K(x,s)=-exp(2*x-5*s/3)/3, f(x)=exp(2*x+1/3), y(x)=exp(2*x), h=0.025)

Example 3. In (13) $K(x,s) = xe^s$, $f(x) = e^{-x}$, $\lambda = 1$. Then the integral equation (13) will take the following form:

$$y(x) - \lambda \int_0^1 xe^s y(s)ds = e^{-x}, x \in [0,1]. \tag{18}$$

We obtain the exact solution of the integral equation (18):

$$y(x) = e^{-x} - x/2.$$

The formulas for calculating the coefficients of the optimal quadrature formula are given below:

$$C_0^{(0)} = (e^h(h^2 - 4h + 6) - 2h - 6)/h,$$

$$C_\beta^{(0)} = e^{h(\beta-1)}(e^{2h}((\beta+1)^2h^2 + (4\beta+4)h + 6) + e^h(-2\beta^2h^2 + +8\beta h - 12) +$$

$$+(\beta-1)^2h^2 - (4\beta-4)h + 6)/h, \quad \beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(0)} = (e^{Nh}(N^2h^2 + 4N - 6) + e^{(N-1)h}((N-1)^2h^2 - 4(N-1)h + 6)) + he)/h,$$

$$C_0^{(1)} = ((h^3 - 6h^2 + 18h - 24)e^h + 6h + 24)/(2h),$$

$$C_\beta^{(1)} = e^{h(\beta-1)}(e^{2h}((\beta+1)^2\ h^3 - (2\beta^2 + 8\beta + 6)h^2 + (12\beta + 18)h -$$

$$-24) + e^h(4\beta^2\ h^2 - 24\beta h + 48) - (\beta-1)^2h^3 - (2\beta^2 + 8\beta + 6)h^2 +$$

$$+(12\beta - 1/)h - 24)/(2h),\ \beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(1)} = (e^{Nh}(N^2\ h^3 + 2Nh2 - (12N-6)h + 24) +$$

$$+e^{(N-1)h}((N-1)^2\ h^3 + 2Nh^2 + (12N-18)h - 24) - 20eh^2 - 4eh)/(h),$$

$$C_0^{(2)} = (e^h(h^4 - 10h^3 + 54h^2 - 168h - 240) - 6h^2 - 72h - 240)/(12h),$$

$$C_\beta^{(2)} = e^{h(\beta-1)}(e^{2h}((\beta+1)^2h^4 - (6\beta^2 + 16\beta + 10)h^3 +$$

$$+(12\beta^2 + 60\beta + 54)h^2 + (-96\beta + 168)h + 240) +$$

$$+e^h(-2\beta^2h^4 + 8\beta h^3 - (12\beta + 12)h^2 + 192\beta h - 480) +$$

$$+(\beta-1)^4 + (6\beta^2 - 16\beta + 10)h^3 + (-6\beta^2 + 30\beta - 18)h^2 +$$

$$+(-96\beta + 168)h + +240)/\,(12h)\,, \quad \beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(2)} = (e^{Nh}(-N^2\ h^4 - (6N^2 - 4N)h^3 - (12N^2 - 36N + 6)h^2 +$$

$$+(96N - 72)h - 240) + e^{(N-1)h}((N-1)^2\ h^4 + (6N^2 - 16N + 10)h^3 +$$

$$+(12N^2 - 60N + 54)h^2 - (96N - 168)h + 240) + he(6N^2\ h^2 + 24Nh + 54))/(12h),$$

$$C_0^{(3)} = (e^h(-h^4 + 12h^3 - 72h^2 + 240h - 360) + 12h^2 + 120h + 360)/(12h),$$

$$C_\beta^{(3)} = e^{h(\beta-1)}e^{2h}(-(\beta+1)^2\, h^4 + (3\beta^2 + 15\beta + 18)h^3 - (12\beta^2 +$$

$$+72\beta + 72)h^2 + (120\beta + 240)h - 360) + e^h(2\beta^2h^4 - 12\beta h^3 +$$

$$+(24\beta^2 + 24)h^2 - 120\beta h + 720) - (\beta-1)^2h^4 - (6\beta^2 - 18\beta + 12)h^3 +$$

$$+(-12\beta^2 + 72\beta - 72h^2 + (120\beta - 240) - 360)/(12h), \quad \beta = 1,\ 2,\ ..., N-1,$$

$$C_N^{(3)} = (e^{(N-1)h}(-(N-1)^2\ h^4 - (6N^2 - 18N + 12)h^3 + (-10N^2 +$$

$$+48N-1)h^2+(120N-240)h-360)+e^{Nh}(N^2h^4+(6N^2-6N)h^3+$$
$$+(12N^2-48N+12)h^2-120(N-1)+360)-he(2N^3\ h^3+12N^2h^2+54Nh+88))/(12h).$$

The results for this example are shown in Table 3a,b,c and Figure 3a,b,c for h = 0.1, 0.05, 0.025.

Table 3a, Results of solving the integral equation (Example 3, K(x,s)=-x*exp(s), f(x)=exp(-x), y(x)=exp(-x)-x/2, h=0.1)

x_i	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	1,0000000	1,0000000	1,0000000	1,0000000	0,00E+00	0,00E+00	0,00E+00	0,00E+00
0,10	0,8548374	0,8547957	0,8548374	0,8548374	0,8548374	4,17E-05	1,39E-08	6,95E-09	3,31E-12
0,20	0,7187308	0,7186474	0,7187308	0,7187308	0,7187308	8,34E-05	2,78E-08	1,39E-08	6,62E-12
0,30	0,5908182	0,5906932	0,5908183	0,5908182	0,5908182	1,25E-04	4,17E-08	2,08E-08	9,92E-12
0,40	0,4703200	0,4701533	0,4703201	0,4703201	0,4703200	1,67E-04	5,56E-08	2,78E-08	1,32E-11
0,50	0,3565307	0,3563223	0,3565307	0,3565307	0,3565307	2,08E-04	6,95E-08	3,47E-08	1,65E-11
0,60	0,2488116	0,2485616	0,2488117	0,2488117	0,2488116	2,50E-04	8,34E-08	4,17E-08	1,98E-11
0,70	0,1465853	0,1462935	0,1465854	0,1465854	0,1465853	2,92E-04	9,73E-08	4,86E-08	2,32E-11
0,80	0,0493290	0,0489955	0,0493291	0,0493290	0,0493290	3,33E-04	1,11E-07	5,56E-08	2,65E-11
0,90	-0,0434303	-0,0438055	-0,0434302	-0,0434303	-0,0434303	3,75E-04	1,25E-07	6,25E-08	2,98E-11
1,00	-0,1321206	-0,1325374	-0,1321204	-0,1321205	-0,1321206	4,17E-04	1,39E-07	6,95E-08	3,31E-11
					Maximum absolute error	**4,17E-04**	**1,39E-07**	**6,95E-08**	**3,31E-11**

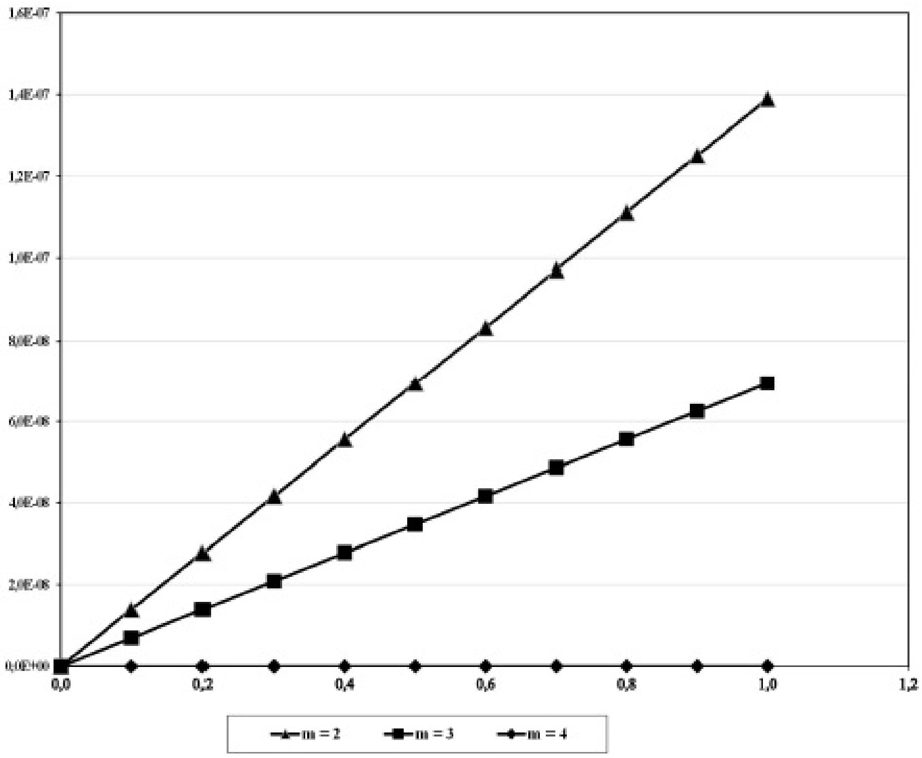

Figure 3a. Graphical illustration of the absolute error (Example 3, K(x,s)=-x*exp(s), f(x)=exp(-x), y(x)=exp(-x)-x/2, h=0.1)

Table 3b, Results of solving the integral equation (Example 3, K(x,s)=-x*exp(s), f(x)=exp(-x), y(x)=exp(-x)-x/2, h=0.05)

x_j	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	1,0000000	1,0000000	1,0000000	1,0000000	0,00E+00	0,00E+00	0,00E+00	0,00E+00
0,10	0,8548374	0,8548270	0,8548374	0,8548374	0,8548374	1,04E-05	8,68E-10	4,34E-10	5,26E-14
0,20	0,7187308	0,7187099	0,7187308	0,7187308	0,7187308	2,08E-05	1,74E-09	8,68E-10	1,04E-13
0,30	0,5908182	0,5907870	0,5908182	0,5908182	0,5908182	3,13E-05	2,60E-09	1,30E-09	1,54E-13
0,40	0,4703200	0,4702784	0,4703200	0,4703200	0,4703200	4,17E-05	3,47E-09	1,74E-09	2,07E-13
0,50	0,3565307	0,3564786	0,3565307	0,3565307	0,3565307	5,21E-05	4,34E-09	2,17E-09	2,58E-13
0,60	0,2488116	0,2487491	0,2488116	0,2488116	0,2488116	6,25E-05	5,21E-09	2,60E-09	3,10E-13
0,70	0,1465853	0,1465124	0,1465853	0,1465853	0,1465853	7,29E-05	6,08E-09	3,04E-09	3,62E-13
0,80	0,0493290	0,0492456	0,0493290	0,0493290	0,0493290	8,33E-05	6,95E-09	3,47E-09	4,13E-13
0,90	-0,0434303	-0,0435241	-0,0434303	-0,0434303	-0,0434303	9,38E-05	7,81E-09	3,91E-09	4,65E-13
1,00	-0,1321206	-0,1322247	-0,1321206	-0,1321206	-0,1321206	1,04E-04	8,68E-09	4,34E-09	5,17E-13
					Maximum absolute error	**1,04E-04**	**8,68E-09**	**4,34E-09**	**5,17E-13**

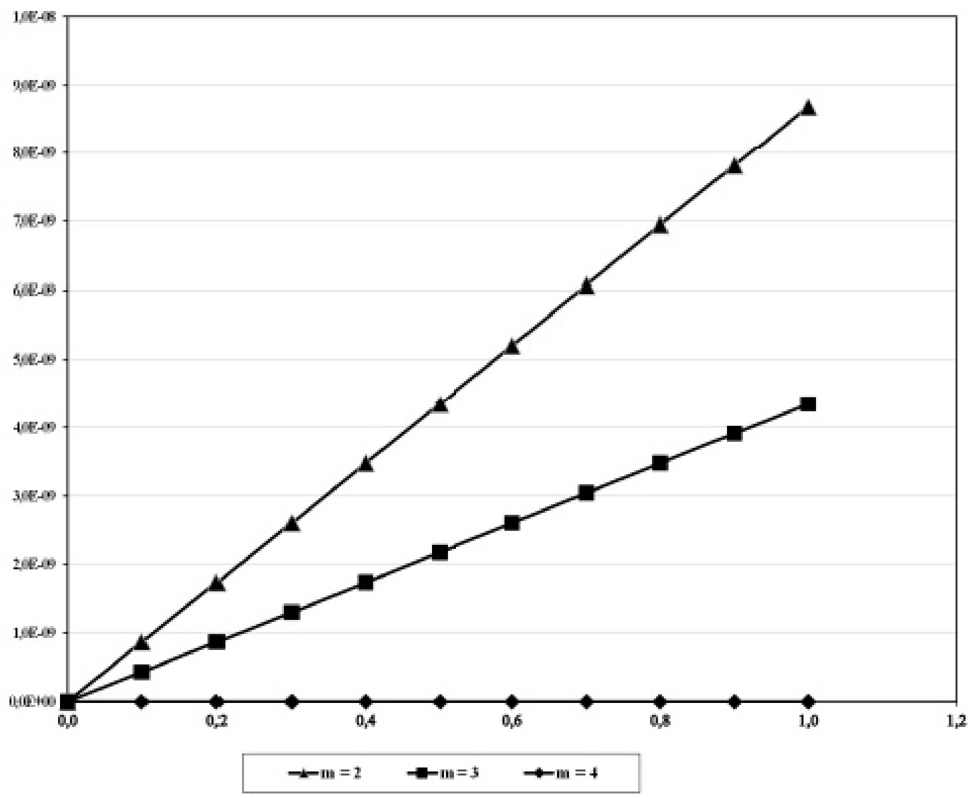

Figure 3b. Graphical illustration of the absolute error (Example 3, K(x,s)=-x*exp(s), f(x)=exp(-x), y(x)=exp(-x)-x/2, h=0.05)

Table 3c, Results of solving the integral equation (Example 3, K(x,s)=-x*exp(s), f(x)=exp(-x), y(x)=exp(-x)-x/2, h=0.025)

x_j	Exact solution	Optimal quadrature formulas				Absolute error			
		m = 1	m = 2	m = 3	m = 4	m = 1	m = 2	m = 3	m = 4
0,00	1,0000000	1,0000000	1,0000000	1,0000000	1,0000000	0,00E+00	0,00E+00	0,00E+00	0,00E+00
0,10	0,8548374	0,8548348	0,8548374	0,8548374	0,8548374	2,60E-06	5,43E-11	2,71E-11	3,00E-15
0,20	0,7187308	0,7187255	0,7187308	0,7187308	0,7187308	5,21E-06	1,09E-10	5,43E-11	7,77E-16
0,30	0,5908182	0,5908104	0,5908182	0,5908182	0,5908182	7,81E-06	1,63E-10	8,14E-11	3,11E-15
0,40	0,4703200	0,4703096	0,4703200	0,4703200	0,4703200	1,04E-05	2,17E-10	1,09E-10	4,55E-15
0,50	0,3565307	0,3565176	0,3565307	0,3565307	0,3565307	1,30E-05	2,71E-10	1,36E-10	3,55E-15
0,60	0,2488116	0,2487960	0,2488116	0,2488116	0,2488116	1,56E-05	3,26E-10	1,63E-10	4,47E-15
0,70	0,1465853	0,1465671	0,1465853	0,1465853	0,1465853	1,82E-05	3,80E-10	1,90E-10	5,55E-15
0,80	0,0493290	0,0493081	0,0493290	0,0493290	0,0493290	2,08E-05	4,34E-10	2,17E-10	8,72E-15
0,90	-0,0434303	-0,0434538	-0,0434303	-0,0434303	-0,0434303	2,34E-05	4,88E-10	2,44E-10	8,98E-15
1,00	-0,1321206	-0,1321466	-0,1321206	-0,1321206	-0,1321206	2,60E-05	5,43E-10	2,71E-10	7,80E-15
					Maximum absolute error	**2,60E-05**	**5,43E-10**	**2,71E-10**	**8,98E-15**

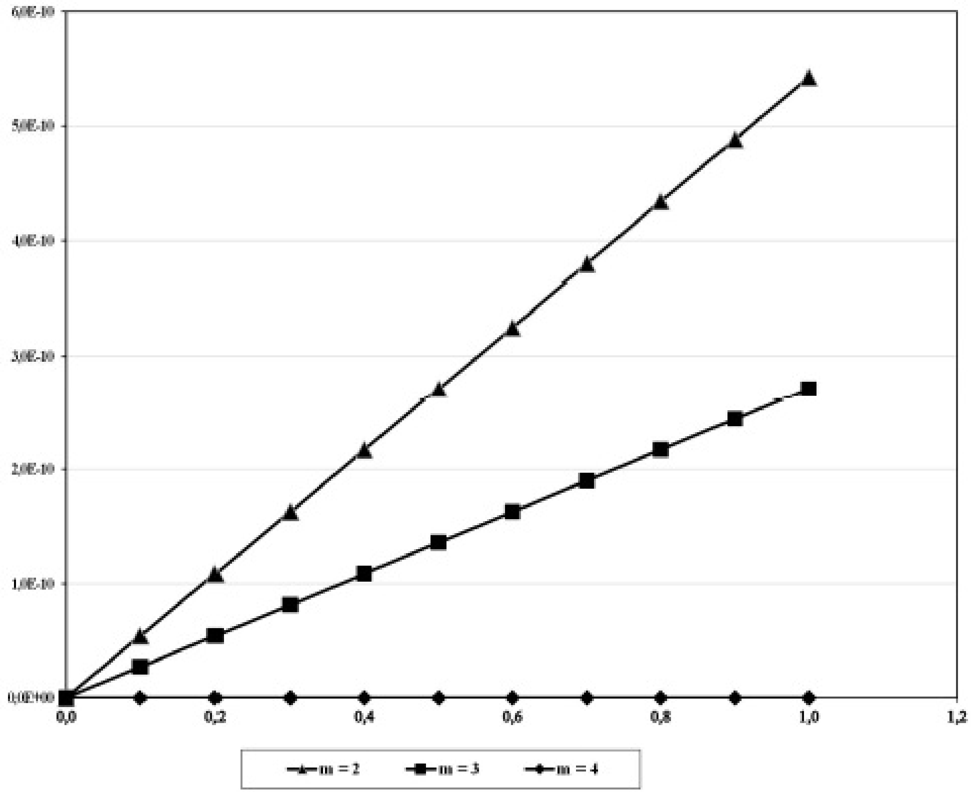

Figure 3c. Graphical illustration of the absolute error (Example 3, K(x,s)=-x*exp(s), f(x)=exp(-x), y(x)=exp(-x)-x/2, h=0.025)

5 Conclusion

The above examples have also been solved by various numerical methods by other authors, namely the Touchard polynomial method [24], the Bernstein polynomial method [23], the Chebyshev polynomial method [21], the Gauss-Lobatto quadrature formula [15], Legendre wavelets [22] and the triangular method functions [9]. The numerical results of the proposed method (PM) are compared with the results of other authors. For comparison, the maximum absolute error was taken, i.e. the maximum value of the absolute value of the difference between the exact solution and the approximate one. The comparison results for different values of h - grid step showed the following:

Example 1: at $h = 0.1$, PM - 1.80E-10, [21] - 0.30E-5;
at $h = 0.05$, PM - 2.81E-12, [21] - 0.30E-5;
at $h = 0.025$, PM - 4.66E-14, [21] - 0.30E-5.

Example 2: at $h = 0.1$, PM - 8.14E-09, [21] - 0.32E-3,
[9] - 6.4E-7, [12] - 1.2E-2, [22] - 5.0E-6;
at $h = 0.05$, PM - 1.27E-10, [21] - 0.32E-3,
[9] 6.4E-7, [12] - 1.2E-2, [22] 5.0E-6;
at $h = 0.025$, PM - 1.98E-12, [21] - 0.32E-3,
[9] - 6.4E-7, [12] - 1.2E-2, [22] - 5.0E-6.

Example 3: at $h = 0.1$, PM - 3.31E-11, [23] - 1.0E-3, [24] - 2.0E-3;
at $h = 0.05$, PM - 5.17E-13, [23] - 1.0E-3, [24] -2.0E-3;
at $h = 0.025$, PM - 8.98E-15, [23] - 1.0E-3, [24] -2.0E-3.

The comparison results prove that the proposed method is better than the methods of other authors.

The results prove that as m increases, the optimal quadrature formulas in the space $L_2^{(m)}(0,1)$ give high accuracy in solving the integral equation.

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ПРИБЛИЖЕННОЕ РЕШЕНИЕ ЛИНЕЙНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ ФРЕДГОЛЬМА ВТОРОГО РОДА МЕТОДОМ ОПТИМАЛЬНЫХ КВАДРАТУР

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В статье рассматривается применение оптимальной квадратурной формулы в пространстве для численного решения линейных интегральных уравнений Фредгольма второго рода. Анализируются результаты конкретных примеров. Точное решение используется для сравнения результатов. Доказано, что с ростом m оптимальные квадратурные формулы в пространстве дают высокую точность решения интегрального уравнения.

Ключевые слова: линейное интегральное уравнение, оптимальная квадратурная формула, коэффициенты оптимальной квадратурной формулы, абсолютная погрешность.

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