UDC 519

NUMERICAL SIMULATION OF UNSTEADY UNDERGROUND WATER FILTRATION IN A POROUS MEDIUM

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The problem of the process of unsteady underground water filtration in a porous medium considered in the paper is an urgent one. It is related to design and development of hydraulic structures, regulation of groundwater runoff, flooding, salinization and waterlogging; all these cause great damage to the national economy. For the development of a mathematical model of the process, the research associated with this problem is analyzed in detail in this paper and a mathematical apparatus for studying and predicting changes in the groundwater level during the process of filtering in porous media is proposed. To construct a mathematical model of the process, Darcy’s law is used and the source of infiltration (rain, watering) and evaporation is taken into account. The task in the study is considered at various boundary and internal conditions. Since the process is described by a nonlinear differential equation in partial derivatives, an analytical solution is difficult to obtain. To solve it a numerical algorithm based on a finite-difference scheme has been developed; an iterative scheme is used for nonlinear terms, its convergence is checked. Various locally-one-dimensional schemes were constructed for solving boundary problems of filtering in simply connected and multiply connected regions. The resulting system of a tridiagonal algebraic equation is solved by the sweep method. To illustrate the developed schemes, several examples of problem solving are given and the results of numerical experiments on computers are analyzed in detail; the conclusions are drawn that the heterogeneity of the aquifer significantly affects the change in the groundwater level during the filtration process. It is found that the weight scheme is the most appropriate to solve the problems of groundwater filtration in multiply connected filtering regions.

Keywords: mathematical model, underground water, groundwater, filtration, numerical algorithm.


1 Introduction

Various hydraulic structures are being built for the need of national economy: dams on rivers to regulate water runoff in hydroelectric power plants, reservoirs and canals. However, the construction of these structures in some cases leads to negative consequences - backwater pressure, flooding, salinization and waterlogging of lands, which causes great damage to the national economy.

Land reclamation activities to fight flooding, salinization, waterlogging require large capital investments. Therefore, the development of effective methods for solving problems of forecasting the change in groundwater level is one of the urgent problems.
Design of powerful and super powerful reservoirs, water intake structures and drainage channels in large gas and oil fields with the highest national economic efficiency requires scientifically based methods for solving such problems; this is also one of the most important tasks of applied mathematics.

Effective tools for integrated research, forecasting and management of the above dynamic processes are the mathematical model, the numerical algorithm and the software to conduct computational experiments on a computer and to make management decisions.

Outstanding scientists deal with the problems in the field of mathematical modeling of the above processes; they have created a number of schools and obtained significant theoretical and applied results.


Significant theoretical and applied results have been obtained by foreign and domestic scientists for the problems of mathematical modeling of groundwater filtration processes in multilayer porous media to predict and regulate and make management decisions.

In particular, in [1], an analysis of fresh water reserves was carried out taking into account economic development and climate changes. The authors evaluated and compared a set of data-driven models, including models of artificial neuron networks (ANN), support vector machines (SVM), and the M5 model tree. The feasibility and capabilities of these models are demonstrated by the example of predicting the level of groundwater five days ahead in the arid and semi-arid basin located in northwestern China. Encouraging results of simulation show that methodologies can simplify and improve the procedure for predicting groundwater levels.

In [2], a mathematical model was used to perform an asymptotic analysis of overpressure fields with filtration consolidation in a double relaxation system; it showed that at the initial stages of consolidation, it is important to take into account the relaxation properties of a deformable porous medium in the case of abrupt and significant changes in pressure. In the general case, the dynamics of filtration consolidation of a porous medium can be numerically simulated within the framework of the mathematical model under consideration based on the developed algorithm.

A mathematical model of the process of salt motion in the filtration transport of salt was developed in [3] taking into account the process of infiltration in unsaturated layered soils. To solve this problem, a solution was obtained by finite-difference method. As a result of the task implementation, numerical experiments were carried out and the obtained results were analyzed.

In mathematical simulation of this process, in [4], the temperature gradients were considered in field conditions and brackish water conditions for the northern semi-arid regions of China. The results of the numerical solution of the problem show that the gradient of soil temperature has a definite effect on the water-salt migration in soil. It was noted that in experiments, the effect of temperature gradient on salt migration was greater than the effect of water movement.

In [5] a two-dimensional steady flow of groundwater in a vertical plane was considered. The analytical solution was developed and used to study the interaction of water with
the surface of groundwater flow. In the paper, the aquifer is idealized as an infinite strip and the channel is modeled as a horizontal equipotential function.

Hydrodynamic and hydraulic models of water runoff in wetlands were proposed in [6], they allow describing filtration and surface runoff processes with varying degrees of detail and accuracy. On the basis of salt transfer models by interacting filtration and channel currents, the issues of modeling the quality of groundwater and surface water were considered.

In [7] the authors dealt with the concept of groundwater filtration. The types and methods of filtration modeling were described. Particular attention is paid to numerical simulation. The role of computer simulation in hydrogeological research was investigated. In addition, the relevance of developing new methods of computer simulation and the creation of a modern information modeling system was emphasized.

The concept of groundwater filtration is discussed in [8]. Types and methods of modeling the filtration of groundwater in multilayer porous media were described. Particular attention was paid to the numerical simulation of the process of filtering groundwater in various modes of movement.

In [9], a mathematical model of the object under study was proposed, in which the generalized Richards equation was used to transport water in unsaturated soil. The resulting mathematical model was integrated by the finite difference method.

The process of aeration zone soaking in various conditions of water inflow to soil was discussed in [10]. At power dependence of water transfer coefficient and linear dependence of the suction height on moisture-content, an analytical function of the initial distribution of moisture-content near the groundwater level was obtained. On the basis of numerical solution of the corresponding initial-boundary value problem, the process of soaking in conditions of close deposit of groundwater was analyzed.

In [11] the use of the finite difference method to solve the problems of forecasting groundwater level changes was discussed by the method of mathematical simulation of anisotropic aquifers on an inclined aquiclade using deterministic models; this simplified the mathematical description of the initial conditions and the modeling process itself, and allowed the use of a standard mathematical apparatus in solving complex problems under non-stationary filtration conditions.

The methods to construct joint models of ground and surface waters were presented in [12], and a computer technology for constructing such systems was given. The technique of modeling the processes of filtration and heat transfer to build models of hydrothermal systems was considered. As examples, the systems of regional models were presented for the conditions of the southeastern part of the West Siberian artesian basin, for the central part of the Novomoskovsk industrial area.

A mathematical model was developed in [13] to calculate a uniform rise in groundwater level in the city of Kharkov; it takes into account additional infiltration into groundwater, precipitation infiltrated into groundwater, transpiration, evaporation and water withdrawal from groundwater. The proposed approach allows the use of numerical assessment methods to predict the process of groundwater level change in built-up areas, taking into account various natural and man-made factors.

In [14] the main approaches to modeling water transfer at different irrigation methods are discussed. The well-known models of water transfer in a porous-capillary medium under stationary and non-stationary mass forces are analyzed with regard to the nonlinear effects that can be applied to subsurface irrigation. In this model, the parameters are: the groundwater level, water loss coefficient, capillary diameters, relative capillary volume,
Numerical simulation of unsteady underground filtration rate, total inflow and outflow, water level in capillaries, capillary rise rate in the capillaries of a certain diameter, capillary rise height, etc. An analysis of discussed mathematical models was conducted.

A mathematical model was developed in [15], to study the distribution of groundwater pressure and its changes in underground structures of cylindrical form. On the basis of the model created, the influence of the aquifer thickness, soil porosity, filtration coefficient, viscosity coefficient and piezoconductivity coefficient was investigated. The analysis of the possibility of pushing the structure and the destruction of the foundation under pressure caused by groundwater was conducted. Analytical formulas were obtained for estimating stresses in the foundation and predicting the possibility of its destruction.

A mathematical model was developed in [16] for predicting groundwater level in two-layer formations. The authors considered a two-layer medium consisting of two layers: soil (with low throughput capacity) and water.

The papers [17-18] deal with numerical simulation of the process of water and salt transfer in soil. A comprehensive study of the proposed mathematical model was, taking into account the pore colmatage (mudding) of soil by fine-dispersed particles over time; changes in coefficient of soil permeability, water loss and filtration; changes in the initial porosity and the porosity of settled mass, as well as an effective numerical algorithm based on the Samarsky-Fryazinov scheme with the second order of approximation. To derive a mathematical model of salt transfer, it was assumed that the pressure gradient in the channel is constant and equal to the atmospheric pressure. The results of calculations for the proposed algorithms were presented in the form of graphical objects, a detailed analysis of these results was given.

A detailed review of initial-boundary problems for non-classical models of mathematical physics generalizing the Cauchy and Showalter-Sidorov conditions was given in [19-21]. The results obtained by authors are illustrated by specific initial-boundary problems for equations and systems of partial differential equations arising in filtration theory, hydrodynamics and mesoscopic theory and considered on sets of various geometric structures.

2 Statement of the problem

To conduct a comprehensive study of the process of unsteady groundwater filtration in a porous medium, proceed to the derivation of a mathematical model of the object under study. Let the layer through which water moves is underlain at the bottom by the aquiclude (Figure 1).

Consider the movement of water with a free surface in such a reservoir. Let us take a certain horizontal plane \(x_0y\) as the reference plane. Assume that the function \(b(x, y)\) is fairly smooth and not changing throughout the reservoir.

The level of the free surface \(H\), measured from the plane \(x_0y\), is a function of the variables \((x, y, t)\). Assume that it also varies little throughout the reservoir. Then the projections on the coordinate axes of the filtration rate can be written in the form:

\[
\begin{align*}
u &= -\gamma k \frac{\partial H}{\partial x}, \\
\gamma &= -\gamma k \frac{\partial H}{\partial y}, \\
w &= 0.
\end{align*}
\] (1)

Where \(\gamma\) is the specific gravity, \(k\) is the filtration coefficient. Considering relation (1), the equation for the level \(H\) is written in the form:

\[
\frac{\partial}{\partial x} [(H - b)u] + \frac{\partial}{\partial y} [(H - b)v] = -\mu \frac{\partial H}{\partial t} + q.
\] (2)
or

\[
\frac{\partial}{\partial x} \left[ (H-b) \frac{\partial H}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (H-b) \frac{\partial H}{\partial y} \right] = \frac{\mu}{\gamma} \frac{\partial H}{\partial t} - \frac{q}{\gamma}. \tag{3}
\]

Here \(\mu\) is the free outflow or lack of saturation.

The more common form is (3), thus the equation for the free surface, or as it is also called for the groundwater level, is a nonlinear second-order partial differential equation. For heterogeneous formations, the filtration coefficient \(k\) is a known function of the variables \((x, y)\). For water it is assumed as \(\gamma = 1\).

Let us dwell on the free term \(q\), which under general assumptions is a function of \((x, y, t, H)\). This feature characterizes infiltration (rain, watering) and evaporation. Numerous field observations show that evaporation depends on the depth of groundwater and can be written as

\[
q = q_0 \left( 1 - \frac{H}{H_{cr}} \right)^n. \tag{4}
\]

Where \(q_0\) is the evaporation from the day surface, \(n\) is the parameter, \(H_{cr}\) is the critical depth. For \(H \leq H_{cr}\) it is assumed that \(q = 0\).

General solution of equation (3) depends on arbitrary functions. Therefore, for a unique solution of this equation, it is necessary to introduce to it the additional conditions. The nature of the additional conditions depends on the type of specific task associated with the study of the behavior of the groundwater level changes.

Let a certain region \(D\) of the plane \(x0y\) simply or multiply connected be given, bounded by a sufficiently smooth curve \(\Gamma\). It is necessary to solve equations (3) within the region \(D\). That is, it is necessary to investigate the groundwater level change for values \((x, y)\) belonging to a region \(D\). This region will be referred to as the filtering region.

One of the additional conditions similar for all types of tasks is the so-called initial condition, which represents the values of the groundwater level at "initial" time \(t_0\) from which \(H(x,y,t)\) should be determined. \(t_0 = 0\), can always be accepted, therefore, the
initial condition is written as

\[ H(x, y, 0) = \varphi(x, y), \quad (x, y) \in D, \]  

(5)

where \( \varphi(x, y) \) is the given and fairly smooth function in \( D \). The remaining additional conditions are set at the boundaries of region \( D \).

The boundary conditions are of the following types:

a) \[ H(x, y, t) = \psi(x, y, t), \quad (x, y) \in, \]  

(6)

where \( \psi(x, y, t) \) is the given function.

Thus, on the boundary of the filtration area for any point in time, the groundwater level values are known.

To solve equation (3) at conditions (5), (6) means to predict the changes in the groundwater level in the filtration area \( D \) at a given initial groundwater level; and when at the boundary of the filtration area the specified (known) level is maintained for the entire investigated period of time.

b) \[ \frac{\partial H}{\partial n} = f(x, y, t), \quad (x, y) \in, \]  

(7)

where \( \frac{\partial H}{\partial n} \) is the normal to the contour, \( \Gamma \) is the derivative, \( n \) is the internal normal to the contour, \( f(x, y, t) \) is the known function.

This type of boundary condition means that a flow is set at the boundary of the filtration area. For example, if \( f = 0 \), the condition (7) means that the boundary \( \Gamma \) is impermeable.

c) condition (6) is set on the part of the boundary, and condition (7) on the other part; in other words, levels are set on one part or parts, and the flow is set on the other or other parts. For a multiply connected region on inner boundaries, any condition of the three types listed above can be specified. If the inner boundary of the multiply-connected filtering area is the contour of the well, then, the following flow rate is set on it

\[ \int (, ) H \frac{\partial H}{\partial n} ds = Q(t). \]  

(8)

where \( \ldots \) is the contour of the well, \( Q(t) \) is the known function (the given flow rate). However, on some wells a condition of the form (6) may be set.

In the theory of mathematical physics, the integration of the equation of mathematical physics in partial derivatives of parabolic type at conditions (5), (6) is called the first boundary value problem, at conditions (5); (7) is called the second boundary value problem, at conditions (5), (6) and (7) - a problem with mixed boundary conditions. It is impossible to construct an analytical solution of these problems for equation (3) in an arbitrary filtering region.

There are a number of papers [22–25] in which, for the simplest forms of the reservoir geometry and taking linearization of equation (3) into account, it is possible to obtain an analytical solution. At the same time, it is theoretically and practically significant to integrate equation (3) under general assumptions, set for the boundary conditions and the reservoir geometry. At present, one of such methods is a numerical method for integrating differential equations based on the finite difference methods.
3 The method of solving the problem of predicting the groundwater level changes and the calculation of vertical drainage using a computer

First consider the solution of the first boundary value problem for equation (3) in a simply connected filtering region $D$, and then separately consider the solution of equation (3) in a multiply connected region $D$ with given flow rate of the wells.

In numerical solution of applied problems, it is advisable to proceed to the dimensionless variables by the formulas

$$H^* = \frac{H}{H_{xap}}, \quad K^* = \frac{K}{K_{xap}}, \quad q^* = \frac{qL^2}{K_{xap}H_{xap}^2},$$

$$\xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad \tau = \frac{K_{xap}H_{xap}}{\mu L^2} t, \quad Q^* = \frac{Q}{H_{xap}^2 K_{xap}}.$$ 

where $H_{xap}$, $K_{xap}$, $\mu$ and $L$ are the characteristic level, filtration coefficient, water loss or a lack of saturation, length, respectively; $\tau$ is the dimensionless time, $Q^*$ is the dimensionless flow rate. The diameter of the region $D$ can be taken as a characteristic length.

In a dimensionless form, equation (3) can be written as:

$$\frac{\partial}{\partial\xi} \left[ K^*(H^* - b^*) \frac{\partial H^*}{\partial\xi} \right] + \frac{\partial}{\partial\eta} \left[ K^*(H^* - b^*) \frac{\partial H^*}{\partial\eta} \right] = \mu^* \frac{\partial H^*}{\partial\tau} - q^*. \quad (9)$$

The initial and boundary conditions are also transformed to dimensionless forms. If $k$ and $\mu$ are constant for the entire filtration area and the aquiclude is horizontal, then equation (9) will be rewritten as:

$$\frac{\partial}{\partial\xi} \left[ (H^* \frac{\partial H^*}{\partial\xi}) \right] + \frac{\partial}{\partial\eta} \left[ (H^* \frac{\partial H^*}{\partial\eta}) \right] = \frac{\partial H^*}{\partial\tau} - q^*. \quad (10)$$

Further, we will deal with the equation in a dimensionless form, therefore, the asterisks in equations (9), (10) are omitted for simplicity of writing. To solve equation (10) by the methods of finite differences, the filtering area $D$, together with the boundary $\Gamma$, is covered with a uniform grid area $\omega_h$, obtained by straight parallel axes $0\xi$ and $0\eta$ carried out with uniform step $h$. Thus, the filtering region $D$ with the boundary $\Gamma$ is assigned to the net domain (Figure 2)

$$\omega_h = \omega^0_h + \omega^*_h + \gamma,$$

where $\omega^*_h$ is a regular set of points for which four crosswise lying adjacent points at the step $h$ are inside the region $D$ or on the boundary $\Gamma$ (Figure 2), $\omega^0_h$ is an irregular set of points (Figure 2) for which at least one of four neighboring points does not belong to the region, $D$, $\gamma_h$ is the set of points of straight lines intersection parallel to $0\xi$ and $0\eta$ axes with boundary $\Gamma$.

For simplicity, assume that the net domain $\omega_h$ covering the filtration region $D$ consists of regular and boundary points. Then

$$\omega_h = \{ (\xi_i = ih, \eta_j = jh); \ i = 0, 1, ..., l_i, \ j = 0, 1, ..., m_j \}$$

Where $l_i$ is the number of divisions on the straight line $\eta_j$; $m_j$ is the number of divisions on the straight line $\xi_i$. Obviously, due to the assumption made above, the points $\xi_0, \xi_{l_i}, \eta_0, \eta_{m_j}$, belong to the boundary points, and all other points are regular.
To solve equation (9) the Samarsky A.A. locally-one-dimensional method [26-27] is applied. According to this method, the level $H$ at $\tau = \tau_{k+1} = (+1)\Delta\tau$, is found by a sequential solution of equations

$$
\mu_\alpha \frac{\partial (\alpha)}{\partial \tau} = L_\alpha(\alpha) + q_\alpha, \\
\tau \leqslant \tau \leqslant \tau_{k+1}, \quad \alpha = 1, 2, \quad q_1 + q_2 = q
$$

(11)

$$
L_1 H = \frac{\partial}{\partial \xi} \left[ K(H - b) \frac{\partial H}{\partial \xi} \right], \\
L_2 H = \frac{\partial}{\partial \eta} \left[ K(H - b) \frac{\partial H}{\partial \eta} \right].
$$

(12)

at initial $H^{(1)} = H(\xi, \eta, \Delta\tau)$; $H^{(2)} = H^{(1)}(\xi, \eta, (+1)\Delta\tau)$; and natural boundary conditions.

Thus, along each line $\eta_j$, the equation (11) is solved with initial conditions known at $\tau = \tau_k$ ($\alpha = 1$). Then, along each straight line $\xi_i$, the equation (11) is solved anew, and the newly found values corresponding to $\tau = \tau_{k+1}$ ($\alpha = 2$) are taken as the initial conditions. The last value will be the solution of equation (9) at $\tau = \tau_{k+1}$ with initial conditions at $\tau = \tau_k$ and natural boundary conditions. Each equation (11) is approximated by a two-layer six-point finite-difference scheme with a weight $\sigma_\alpha$ as follows

$$
\frac{\mu_{\alpha,k+1} H^{(\alpha)k+1} - H^{(\alpha)}}{\Delta\tau} = \Lambda_\alpha \left[ \sigma_\alpha H^{(\alpha)k+1} + (1 - \sigma_\alpha)H^{(\alpha)k} \right] + \tilde{q}_\alpha, \\
(\alpha = 1, 2)
$$

(13)
here
\[ L_\alpha \sim \Lambda_\alpha, \quad q_\alpha \sim \bar{q}_\alpha, \quad \frac{\partial H^{(\alpha)}}{\partial \tau} = \frac{H^{(\alpha)k+1} - H^{(\alpha)}}{\Delta \tau}, \]
\[
\Lambda_1(H) = \frac{B_{i+0,5,j,k}H_{i+1,j,k} - (B_{i+0,5,j,k} + B_{i-0,5,j,k})H_{i,j,k} + B_{i-0,5,j,k}H_{i-1,j,k}}{h^2}, \tag{14}
\]
\[
\Lambda_2(H) = \frac{B_{i,j+0,5,k}H_{i,j+1,k} - (B_{i,j+0,5,k} + B_{i,j-0,5,j,k})H_{i,j,k} + B_{i,j-0,5,j,k}H_{i,j-1,k}}{h^2}. \tag{15}
\]

Here \( B_{i,j,k} = K_{i,j}(H_{i,j,k} - b_{i,j}), \) \( H_{i,j,k} = (H_{ih}, jh, k\Delta \tau). \)

The sequence of schemes (13) can be conditionally denoted as \( \Lambda_1^{(\sigma_1)} \rightarrow \Lambda_2^{(\sigma_2)}. \) For the solution of (14), the sweep method with iteration is used.

In \( \Lambda_n(H_{k+1}), \) \( B_{i,j,k} \) is replaced by \( B_{i,j,k+1}, \) then the scheme (13) becomes linear; applying to it the sweep method, the solution to the problem is found at \( \tau = \tau_{k+1}. \) Substituting the found value \( H_{i,j,k+1} \) into (15), the solution to equation (13) is found anew for the same \( \tau = \tau_{k+1} \), etc. This process will continue until the following ratio is satisfied.

\[
\max_{i,j \in W_h} \left| H^{(s+1)}_{i,j+1} - H^{(s)}_{i,j+1} \right| < \varepsilon, \quad \varepsilon > 0.
\]

Write down the formulas by which at each iteration step at the moment \( \tau = \tau_{k+1} \) the groundwater levels are determined at the nodes of the net domain \( W_h. \)

Equation (13) is equivalent to the equations
\[
H^{(s+1)}_{(1),i,j,k+1} = A_{i+1,j}H^{(s+1)}_{(1),i+1,j,k+1} + D_{i+1,j}, \tag{16}
\]
\[
H^{(s+1)}_{(2),i,j,k+1} = C_{i,j+1}H^{(s+1)}_{(2),i,j+1,k+1} + E_{i,j+1}. \tag{17}
\]

where
\[
A_{i+1,j} = \frac{\sigma_1 B_{i+0,5,j,k}}{h^2} + \sigma_1(B_{i+0,5,j,k} + B_{i-0,5,j,k}) - \sigma_1 B_{i-0,5,j,k}A_{i,j}, \tag{18}
\]
\[
D_{i+1,j} = \frac{\sigma_1 B_{i-0,5,j,k}}{h^2} + \frac{\sigma_1 B_{i+0,5,j,k}}{h^2} + \frac{(1 - \sigma_1)\Lambda_1(H_{1}(1))}{h^2} - \sigma_1 B_{i-0,5,j,k}A_{i,j}, \tag{19}
\]
\[
C_{i,j+1} = \frac{\sigma_2 B_{i+0,5,k}}{h^2} + \frac{\sigma_2 B_{i+0,5,k} + B_{i+0,5,k}}{h^2} - \sigma_2 B_{i+0,5,k}C_{i,j}, \tag{20}
\]
\[
E_{i,j+1} = \frac{\sigma_2 B_{i-0,5,j,k}}{h^2} + \frac{\sigma_2 B_{i+0,5,j,k}}{h^2} + \frac{(1 - \sigma_2)\Lambda_2(H_{2}(2))}{h^2} - \sigma_2 B_{i+0,5,j,k}C_{i,j}. \tag{21}
\]

From boundary conditions at \( i = 0, \) along each straight line \( \eta_j \) and \( j = 0, \) along each straight line \( \xi_i, \) it follows that \( A_{1,j}; D_{1,j}; C_{1,i}; E_{i,1} \) are known values.

Consequently, \( A_{i,j}; D_{i,j}; C_{i,j} \) and \( E_{i,j} \) are determined by formulas (18) - (21). Further, from the boundary conditions at \( i = l_i, \) along each \( \eta_j \) and \( j = m_j \) and along each \( \xi_i, \) and from (16) and (17) the following values are calculated sequentially
\[
H^{(s+1)}_{l_i,j-1,j+1}, \quad H^{(s+1)}_{l_i,j+1}, \quad H^{(s+1)}_{i,m_j-1,j+1}, \quad H^{(s+1)}_{i,j+1}.
\]

In similar way the boundary-value problems are solved in a multiply-connected domain by a locally one-dimensional method; however when solving the problem of calculating vertical drains or well interference, along with general issues of efficient approximation of a differential operator, choosing a method for solving finite-difference equations, rational
distribution of computer RAM and saving machine time, there arises the problem of approximation of conditions on wells.

This is explained by the fact that the approximation of the well contour by practically acceptable uniform grid is impossible. Application of a grid with variable pitch for the class of problems in question is not effective. Therefore, studies were aimed at finding out the possibility of approximation of the well contour by a nodal point. It turned out that the results correspond to a problem with some enlarged well of a radius $R_\Phi$ commensurate with the grid step and satisfying the ratio $R_\Phi \approx 0.2h$ (Figure 3).

As seen the grid step determines the radius of the enlarged well. Knowing the values of the sought for function $H(\xi, \eta, \tau)$ on the contour of an image well and considering these values as boundary conditions, we can further solve the problem of determining $H(\xi, \eta, \tau)$ on the wall of real well $R_c$ and in its vicinity. The task is to integrate (9) for axisymmetric filtering; the conditions of the form (8) or (6) are given on the wells.

To approximate the boundary conditions on the wells by finite-difference relations, proceed as follows. Select the volume element with the axis of symmetry passing through a special point (well). Consider the balance equation for this volume, the horizontal projection of which is a square with a side $h$. Summing up the values of flow through the side faces in the direction of the axes $o\xi$ and $o\eta$, we obtain the expression

$$(H_{i-0.5,j} - b_{i-0.5,j})V_{i-0.5,j}h - (H_{i+0.5,j} - b_{i+0.5,j})V_{i+0.5,j}h + (H_{i,j-0.5} - b_{i,j-0.5})V_{i,j-0.5}h - (H_{i,j+0.5} - b_{i,j+0.5})V_{i,j+0.5}h$$

where $V$ is the filtration rate. This expression for an ordinary node must be equal to

$$-\frac{\Delta H_{cp}}{\Delta \tau}h^2$$

Here $H_{cp}$ is the average value of function $H(\xi, \eta, \tau)$ in the domain $\xi_{i-0.5} \leq \xi \leq \xi_{i+0.5}$, $\eta_{j-0.5} \leq \eta \leq \eta_{j+0.5}$. 
For a particular node, a takeout at the well $Q$ is added to this value and in conditions of infiltration or evaporation a value $q_i$ is added. Thus, for the filtration obeying Darcy’s law, a locally-one-dimensional A.A. Samarsky equation is obtained in the form

$$
\frac{\mu_{(a)+1} H_{(a)+1} - H_{(a)}}{\Delta \tau} = \Lambda_a \left[ \sigma_a H_{(a)+1} + (1 - \sigma_a) H_{(a)} \right] Q_{(a)},
$$

$$
Q_{(a)} = Q + q_{a}.
$$

It is easy to see that the scheme (22) is a generalization of (14) and the differential equation (9) is obtained by limit transfer at $\sigma_a = 1$. Solution (22) makes it possible to obtain $H(\xi, \eta, \tau)$ in the grid points and, therefore, it is possible to determine the groundwater level on the grid of the well of a radius $R$. It is obvious that solution (22) will be approximate since the takeout from an image well is not known. Therefore, assuming in a zero approximation that $Q = Q$, the takeout from an image well can be obtained

$$
Q = 2\pi R(R) \left[ H(R, \tau) - b(R, \tau) \right] \frac{\partial H(R, z)}{\partial z}.
$$

(23)

Further, the process of finding a solution is repeated until the condition is met:

$$
\max_{i,j \in w} \left| H_{i,j+1}^{(s+1)} - H_{i,j+1}^{(s)} \right| < \varepsilon, \varepsilon > 0.
$$

Various locally-one-dimensional schemes were constructed to solve the boundary problems of filtering in simply and multiply connected domains. Some problems obtained by various locally-one-dimensional schemes were compared with practical exact (numerical) solutions available for these problems.

As shown by the calculations, the diagram of $\Lambda_1^{(\sigma_1)} \rightarrow \Lambda_2^{(\sigma_2)}$, give good results for simply-connected areas corresponding to the problems of predicting the groundwater level under the effect of hydraulic structures located outside the filtration area.

When solving the problems on the interference of wells, it turned out that locally-one-dimensional schemes have a high accuracy when symmetrized as follows

$$
0, 5\Lambda_1^{(\sigma_1)} \rightarrow 0, 5\Lambda_2^{(\sigma_2)} \rightarrow 0, 5\Lambda_2^{(\sigma_2)} \rightarrow 0, 5\Lambda_1^{(\sigma_1)},
$$

(24)

Even more accurate results are obtained by summation with some weight $\chi$ of solutions obtained independently of each other according to the schemes

$$
\Lambda_1^{(\sigma_1)} \rightarrow \Lambda_2^{(\sigma_2)}
$$

and

$$
\Lambda_2^{(\sigma_2)} \rightarrow \Lambda_1^{(\sigma_1)}.
$$

This can be schematically presented as

$$
\chi(\Lambda_1^{(\sigma_1)} \rightarrow \Lambda_2^{(\sigma_2)}) + (1 - \lambda)(\Lambda_2^{(\sigma_2)} \rightarrow \Lambda_1^{(\sigma_1)}).
$$

(25)

To determine the level of wells, when the flow rate is set on it, proceed as follows. Since, as a result of solving equation (22), the value of the level on the contour of image wells of radius $R_\Phi$ becomes known, the level on real wells of radius $R_c$ can be determined as follows. If to neglect the influence of the unsteady nature of the ring flow $R_c \leq z \leq R$ for $\tau \leq \tau \leq \tau_{+1}$ (Fig.3), then from Dupuis formula we get:

$$
H_c = (H^2 - Q_c \ln R_c)^{1/2}.
$$
If it is necessary to take into account the influence of the unsteady nature of the flow, then \( H \) is obtained as a result of equation solution

\[
\frac{\partial H}{\partial t} = \frac{\partial}{\partial z} \left( z \frac{\partial H}{\partial z} \right).
\]

In a ring restricted by circumferences \( R_c \) and \( R_\Phi \) at boundary conditions

\[
\frac{\partial H^2(R_c, t)}{\partial z} = \frac{Q_c}{\pi R_c},
\]

\[
H(R, t) = \tilde{H}(t),
\]

\[
H(z, 0) = H_0.
\]

where \( \tilde{H} \) is the solution of planned unsteady filtering with image well radii.

For the illustration consider a few examples solved on a computer, according to the scheme described above.

**Problem 1.** The unsteady groundwater inflow to the central well of radius \( R_c \), working with a fixed rate of flow \( Q \), in a homogeneous porous medium, circular in plan, of radius \( R \).

The initial condition is characterized by a constant value of the groundwater level

\[
H(x, y, 0) = H_0 = \text{const},
\]

\[
(x, y) \in R_c \leq \sqrt{x^2 + y^2} \leq R,
\]

Boundary conditions are:

On the well

\[
\int_\gamma K H \frac{\partial H}{\partial n} d\gamma = Q,
\]

\( \gamma \) - is the well contour (circumference of radius \( R_c \)). At the border of the filtration area \( \Gamma \), i.e. circumference of the radius \( R \), the boundary conditions are set in two forms

\[
a) = H(x, y, t) = H_0, \quad t > 0,
\]

\[
(x, y) \in \sqrt{x^2 + y^2} \leq R_k
\]

\[
b) \frac{\partial H(x, y, t)}{\partial n} = 0, \quad t > 0, \quad x, y \in \sqrt{x^2 + y^2} \leq R.
\]

For the case a) an unsteady filtering stabilizes rather quickly and the regime of groundwater movement is established; in the case b) there is a depletion of the aquifer with a fixed rate of flow \( Q \).

The problems are solved according to locally-one-dimensional schemes

\[
A^{(\sigma_1)}_1 \rightarrow A^{(\sigma_2)}_2,
\]

\[
0, 5A_1^{(1)} \rightarrow 0, 5A_2^{(1)} \rightarrow 0, 5A_2^{(1)} \rightarrow 0, 5A_1^{(1)};
\]

\[
0, 5A_1^{(1)} \rightarrow 0, 5A_2^{(1)} + 0, 5A_2^{(1)} \rightarrow 0, 5A_1^{(1)};
\]

\[
0, 5A_2^{(0)} \rightarrow 0, 5A_1^{(1)} + 0, 5A_2^{(0)} \rightarrow 0, 5A_2^{(1)}.
\]

at \( Q(1) = 0, Q(2) = 0, 5Q, Q(3) = 0, 5Q, Q(4) = 0 \).
The solution of the problems under consideration can also be obtained by integrating the differential equation

\[
\frac{1}{z} \frac{\partial H}{\partial z} \left( z \frac{\partial H}{\partial z} \right) = \frac{2\mu}{K} \frac{\partial H}{\partial t},
\]

(33)

At conditions

\[ Q = 2\pi R_c H_c \frac{\partial H(R_c, z)}{\partial z}, \quad t > 0, \]

(34)

a) \[ H(R_c, t) = H_{xap}, \quad t > 0, \]

(35)

b) \[ \frac{\partial H(R, z)}{\partial z} = 0, \quad t > 0, \]

(36)

\[ H(z, 0) = H_{xap}, \quad R_c \leq z \leq R. \]

(37)

The solution to this problem can be obtained with any accuracy, using the sweep method with iteration. In order to properly account for the behavior of the solution in the vicinity of the wells, it is necessary to move from variable \( z \) to a dimensionless variable \( u = \ln \frac{z}{R} \).

Then if to introduce the dimensionless variables to the formulas

\[
h = \frac{H}{H_{xap}}, \quad \tau = \frac{H_{xap}K}{\mu R^2} t, \quad Q' = \frac{Q}{\pi K H_{xap}^2},
\]

the problem (34)-(38) is transformed into

\[
\frac{1}{h} \frac{\partial h^2}{\partial \tau} = e^{-2u} \frac{\partial^2 h^2}{\partial u^2},
\]

(38)

\[
h(z, 0) = 1,
\]

(39)

\[
\frac{\partial h^2(u_c, t)}{\partial u} = Q',
\]

(40)

\[
h(0, t) = 1,
\]

(41)

\[
\frac{\partial h(0, t)}{\partial u} = 0.
\]

(42)

For the solution of (38) - (42), the following finite difference scheme can be related

\[
h^2_i = A_{i+1} h^2_{i+1} + B_{i+1},
\]

(43)

where

\[
A_{i+1} = \frac{1}{2 - i} + \frac{1}{\gamma h^2_{i+1}}, \quad B_{i+1} = A_{i+1} (B_i + \frac{e^{2u_i} h^2_{i+1}}{\gamma h^2_{i+1}}),
\]

\[
A_1 = \frac{3}{2} - \frac{e^{2u_1}}{2\gamma h^2_{1}}, \quad B_1 = \frac{e^{2u_1} h^2_1}{2\gamma h^2_{1}} - \Delta u Q'.
\]

Then after determining \( A_1, A_2, A_3, \ldots, A_{n-1}, B_1, B_2, B_3, \ldots, B_{n-1} \) from (42) with condition (39) or (41) \( h_{i,k}, \quad i = n, \quad n - 1, \ldots, 1, 0 \) is determined.

It is obvious that \( h_{i,k} \) at each time step should be integrated until the condition is met

\[
\max_i \left| h_i^{(s)} - h_i^{(s-1)} \right| < \varepsilon, \quad \varepsilon > 0.
\]
Note that, using this problem as an example, a comparison was made of the value $H$ at the boundary of a fictitious well obtained locally by a one-dimensional method with the replacement of a real well with a fictitious one and with a flat radial inflow to the well equal to the fictitious well radius.

The schemes (31) and (32) turned out to be effective for this problem. For the condition of the form (29) or (36), the deviation of the solution obtained by these schemes from the practical exact one at time when 50% of the water reserves are taken out does not exceed 1%. For problems (28) or (35) at time when the process becomes steady, the results coincide with the accuracy of the calculation error.

The computational experiment was carried out when solving planned and flat radial problems. From a comparison of the numerical calculations performed, it is seen that they completely coincide with each other. It is accepted that $R_c = 10 \text{ sm}$, $k = 5 \text{ m/ day}$, $\mu = 0.25$, $Q = 4000 \text{ m}^3/\text{day}$. $R_k = 2500 \text{ m}$.

The same numerical calculations were carried out under the condition that $\frac{\partial H}{\partial n} = 0$ on the boundary of filtration area.

**Problem 2.** Now consider the problem of the inflow of groundwater in a circular filtration area bounded by a circumference of radius $R_k = 2500 \text{ m}$ at a fixed filtration coefficient $k = 5 \text{ m/ day}$ to a battery of four wells located concentrically at a distance of 100m from the center of the filtering area. The flow rate of each well is $Q_j = 4000 \text{ m}^3/\text{day}$, $(j = 1, 4)$. The condition $\frac{\partial H}{\partial n} = 0$, $t > 0$, is maintained at the border of the region $\Gamma$.

The initial condition is characterized by a fixed value of the groundwater level.

$$H(x, y, 0) = 250, \quad (x, y) \in D.$$  

Due to the symmetry, this problem should be symmetrical, i.e. identical, to each of the four squares; this is confirmed by the calculations carried out on a computer.

The solution to this problem is obtained in the form of the lines of different groundwater levels at time $t = 1080 \text{ day}$. Due to the full symmetry of the groundwater movement pattern only a fourth of the filtration area is shown (Figure 4).

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**Figure 4** Change in groundwater level over time
The effect of aquifer heterogeneity is clearly seen in the distribution of lines of equal groundwater level.

**Problem 3.** Consider the problem of interaction of two wells with equal production rate in a rectangular area $ABCD$ (Figure 6); the distance between the wells is 1000 meters. $AD = 2300m$. $AB = 1600m$. The flow rate of one well is $Q = 4000m^3/day$.

Boundary conditions:

$$H(x, y, t)|_{AB} = H(x, y, t)|_{CD} = 250 m,$$

(44)

$$\frac{H(x, y, t)}{\partial n}|_{BC} = \frac{H(x, y, t)}{\partial n}|_{AD} = 0.$$  

(45)

Initial conditions:

$$H(x, y, 0) = 250 m,$$

(46)

$$ (x, y) \in D + \Gamma.$$  

Figure 5 Change in groundwater level at two wells with the same flow rates

A symmetrical arrangement of the lines of equal groundwater levels in the left and right parts of the rectangular area was obtained, which was to be expected. The solution to this problem is obtained according to scheme (26).

In cases when the filtering area has a discontinuous filtration coefficient $(x, y)$, the following problem is solved. The filtration coefficient in the area $ABB'A'$ (the left part of the filtration area) is equal to $k = 5 m/day$. , in the area $A'B'D$ to, $= 8 m/day$. (Figure 6). The remaining conditions correspond to (45), (46), (47). The effect of heterogeneity of the aquifer is clearly seen on the distribution of the lines of equal groundwater levels. As seen from the computer numerical experiments and the isolines shown in Fig. 6, the groundwater level decrease is more noticeable in the right side of the filtration area, which has the best collector properties.

Thus, a locally-one-dimensional scheme allows solving the problems of unsteady filtering with account of details in plan, water-pressure gradient, infiltration feeding and evaporation. Here, the scheme (25) is the most acceptable for solving problems in a multiply connected region.
4 Conclusions

As follows from the carried out numerical calculations, the scheme of the type $\Lambda^{(\sigma_1)}_1 \rightarrow \Lambda^{(\sigma_2)}_2$ gives good results for a simply-connected region to solve the problem of predicting the groundwater level under the effect of hydrotechnical structures.

As seen from numerical calculations on computer, the heterogeneity of the aquifer significantly affects the change in the groundwater level during the filtration process. Computational experiments conducted on a computer, at different values of filtration capacity showed that with increasing value of this parameter, the groundwater level decreases along the length.

Computational experiments have established that for solving the problems of groundwater filtering in a multiply connected region, the most acceptable is the weight scheme (25).

References


Рассматриваемая в статье проблема процесса нестационарной фильтрации подземных вод в пористой среде является актуальной. Это связано с проектированием и разработкой гидroteхнических сооружений, регулированием стока подземных вод, затоплением, засолением и заболачиванием; Все это наносит большой ущерб народному хозяйству. Для разработки математической модели процесса исследования, связанные с этой проблемой, подробно анализируются в этой статье, и предлагается математический аппарат для изучения и прогнозирования изменений уровня подземных вод в процессе фильтрации в пористых средах. Для построения математической модели процесса используется закон Дарси и учитывается источник инфильтрации (дождь, полив) и испарения. Задание в исследовании рассматривается при различных граничных и внутренних условиях. Поскольку процесс описывается нелинейным дифференциальным уравнением в частных производных, аналитическое решение получить сложно. Для ее решения был разработан численный алгоритм на основе конечно-разностной схемы; для нелинейных членов используется итерационная схема, проверяется ее сходимость. Для решения краевых задач фильтрации в односвязных и многосвязных областях построены различные локально-одномерные схемы. Полученная система трехдиагонального алгебраического уравнения решается методом развертки. Для иллюстрации разработанных схем приведено несколько примеров решения задач и детально проанализированы результаты численных экспериментов на компьютерах; Сделаны выводы, что неоднородность водоносного
горизонта существенно влияет на изменение уровня грунтовых вод в процессе фильтрации. Установлено, что весовая схема является наиболее подходящей для решения задач фильтрации подземных вод в многосвязных регионах фильтрации.

**Ключевые слова:** математическая модель, подземные воды, подземные воды, фильтрация, численный алгоритм.